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From Mathmagicians To Mathematicians: Constructing Cognitive Understanding Over Trusting The Algorithm

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FROM MATHMAGICIANS TO MATHEMATICIANS:
CONSTRUCTING COGNITIVE UNDERSTANDING
OVER TRUSTING THE ALGORITHM

by

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A capstone submitted in partial fulfillment of the requirements for the degree of Master of Arts in Teaching.

Hamline University
Saint Paul, Minnesota
May 2017

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To my parents, Mary and Rick, for their unwavering support, and for reminding me that when I grow up I want to be a teacher.
# TABLE OF CONTENTS

CHAPTER ONE: Introduction .............................................................................. 1

How Will This Be Accomplished ................................................................. 2

CHAPTER TWO: Literature Review ................................................................. 8

Constructivism ......................................................................................... 9

General Trajectory of Learning Multiplication ......................................... 12

The Role of the Teacher ......................................................................... 19

The Trouble With The Algorithm .............................................................. 23

Opportunities From Problem Solving ......................................................... 27

Conclusion ............................................................................................... 31

CHAPTER THREE: Design Process .............................................................. 33

Setting and Audience ............................................................................. 35

Best Practices and “System of Instruction” as Building Blocks for Curriculum

Design ..................................................................................................... 36

Conclusion ............................................................................................... 41

CHAPTER FOUR: Project Design ................................................................. 42

State & National Standards .................................................................... 43

Pre-assessment & Formative Assessment ............................................... 44

Daily Lesson Frame ................................................................................ 47
Anticipated Movement Through Taxonomy Within the Large Group Lessons and Resulting Discussions………………………………………………..53
Conclusion…………………………………………………………………………….60
CHAPTER FIVE: Conclusions……………………………………………………63
Reflections………………………………………………………………………….63
Limitations…………………………………………………………………………66
Future Steps……………………………………………………………………….68
REFERENCES……………………………………………………………………69
APPENDIX……………………………………………………………………..74
TABLE OF FIGURES

| Figure 2.1 | Multiplication trajectory | 13 |
| Figure 2.2 | Equal groups | 13 |
| Figure 2.3 | Repeated additions | 14 |
| Figure 2.4 | Skip counting | 14 |
| Figure 2.5 | Multiplicative strategies | 15 |
| Figure 2.6 | Strategy progression | 16 |
| Figure 2.7 | Hiebert’s flow of teacher tasks | 22 |
| Figure 4.1 | Example pre-assessment questions | 46 |
| Figure 4.2 | Daily lesson frame | 48 |
| Figure 4.3 | Example of projected work | 51 |
| Figure 4.4 | Real life models of arrays | 55 |
| Figure 4.5 | Representational models of arrays | 55 |
| Figure 4.6 | Count all compared to count by | 56 |
| Master Grid | 58 |
CHAPTER ONE

Introduction

Math and I got along fairly well until eighth grade, when Mrs. Singer and I disagreed on how algebra should work. I can’t recall exactly what I struggled with, but it was soon made clear that there was one way to do it, the “right” way, and you must show your work. Prior to eighth grade math made a lot of sense to me, with the exception of fractions, but in time and with practice those also made sense. Counting on a number line, using blocks and beans to add and subtract, memorizing the multiplication table, working through long division problems, all seemed possible, if not a little tedious. In general though, math came easy to me.

What I remember about learning math in elementary school was a lot of memorization and practice. I am sure that my teachers explained how to do problems, but I don’t remember being taught theory or reasoning behind the math, and honestly I don’t know if that would have been helpful. What I think is nice about math is that there are very real world practical applications for most of it, and a student can see that, it helps to make it tangible. However, so often today students are taught and expected to memorize the standard algorithms for mathematical operations. Phrases like “carry the 1” or
“borrow from the 7” are often part of the language explaining how to add and subtract. But what does that even mean? And how do those tricks lead to understanding the fundamentals of mathematical operations? Students are not given enough time or space to explore the foundations of mathematical understanding but are instead taught one or two methods for completing a problem that they might not fully understand. This can lead to frustration and misunderstandings further down their mathematical paths and leave them with little understanding for number sense, or flexible thinking about math and logic.

So what if instead of teaching the algorithms with the tricks and steps they entail, teachers were to act as guides to help students develop this flexible thinking? Give them the language to describe the big ideas and fundamentals of math that will better serve them when faced with math challenges in the future, and help them to develop their own problem solving strategies. This paper will examine what happens when students are given opportunities to explore fundamentals of mathematics, and how they develop effective problem solving strategies. Developing a framework of study that offers students a chance to explore strategies and build on their own understanding. This result of this shift from traditional instruction to a more student centered approach, and student successes within it will hopefully provide insight to whether Cognitively Guided Instruction leads to more flexible logico-mathematical thinking (Carpenter et al., 2015).

**How Will This Be Accomplished?**

The goal of this capstone is to determine what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade. One of the foundations of
the following proposed framework incorporates relevant and engaging word problems. The challenge word problems present are two-fold. First students have to figure out what the question is asking and come up with a strategy or a drawing that helps them to determine what they need to know and then they have to do the math to get the answer. The words and context around the “equation” help to give meaning to the actions taking place. Transitioning from student to teacher to now, a teacher-researcher, I need to remember what helped me learn math, and that is real world application. Creating opportunities that allow students to create authentic learning experiences that they can incorporate into their mathematical understandings. I won’t follow in the footsteps of Mrs. Singer and claim there is one “right” way and demand that each student show the same work. I need to diverge from that school of thought and create a space for students to stretch their own thinking and develop stronger, more internalized understanding of mathematical concepts.

It is a big deal to abandon how I was taught, the memorization and drills, the blind trusting of a tricky algorithm. While studying methodology for teaching mathematics I was introduced to Carpenter and Fennema’s work with Cognitively Guided Instruction or CGI (2015). CGI starts with growing math skills from where the students are at in their own understanding. By scaffolding problems and discussions around math students are able to develop strategies and more flexible thinking. This process can be introduced to school aged children but it can also start early on for children as young as two or three when they are developing their understanding of numbers as symbols for quantities of things, two bunnies, 1-2-3-4 crackers. It then begins to introduce big ideas of math,
through exploring operations. This can be done using thoughtfully planned and developmentally appropriate story problems and manipulatives such as beads, blocks or chips. Primary students can work through real-life problems of addition, subtraction and even division through equal parts sharing problems (Carpenter et al., 2015).

An equal parts sharing problem could be;

*Arwyn wanted to share cookies with her 6 friends. She had 24 cookies to give out.*

*She wanted each friend to have the same amount of cookies. How many cookies could she give each friend?*

An addition story problem might look like;

*Mary had 14 Pokémon cards and for her birthday she received 12 more. How many Pokémon cards does have Mary have now?*

Problems like this give students a real-life situation that calls for a mathematical operation. They can imagine the scenario; they can use manipulatives to work it out and they can come to a reasonable solution using their own strategies.

Later, as children develop and grasp the big ideas, the problems can become more complex. Second and third grade students can work with multiplication, division and fractions as they work through problems using drawings, hash marks or tallies, and stretch their number sense understanding to develop strategies for working with larger and more challenging problems. Once students develop their number sense and begin storing this understanding, they can build on it more quickly. Teachers can guide students to use their independently constructed number sense and understanding so that they can become more efficient with their skills. They will practice their mental math to develop
their own strategies for getting to answers, and most importantly, be able to explain how they got there.

When asking a student to divide 284 into 4 equal groups:

Consider this conversation:

Teacher: How many times does 4 go into 2?

Student: It doesn’t.

Teacher: Ok, how about 28?

Student: Yes, 7 times. Because 4 times 7 is 28.

Now consider this conversation:

Teacher: How many times does 4 go in to 200? or 280?

Student: Oh, I know that 4 goes into 200, 50 times.

Teacher: How many times does 4 go into 80?

Student: 80? You can get 4 into 80 20 times.

Teacher: Okay, so what do you with the 50 and the 20?

Student: Add them up, so 50+20 is 70 times, and then there is just 4 left over, and that’s 1 time each, so the answer is 71.

The second conversation and resulting explanation shows that they not only have number sense, but they understand the fundamentals of division, not just that they memorized a procedure. The biggest issue that I have seen as a result of trusting the algorithm is that students do not recognize when they’ve made a glaring error, because they assume if they followed the procedure, they’ll get the right answer. One area of understanding where this is often confused or misused is place value and making sure
that every digit is in its proper place. Take the 284 divided by 4 example. If the student, attempting the standard algorithm, thought Oh, I know 4x7 is 28 so 28/4 is 7 and they placed the 7 over the 2, they would get a very different answer, because the standard algorithms do not promote keeping place value at the forefront of thinking when following procedures. For another example of this think about subtraction. When you “borrow” from the number to the left of another number, what are you really doing? How is place value preserved in this procedure? Developing the language around fundamentals of mathematics, like place value and operations is a cornerstone of Cognitively Guided Instruction. When students have the ability to explain their thinking, teachers have a better ability to guide them to bigger challenges, allowing for deeper understanding.

Developing number sense and allowing space and time for flexible problem solving takes time and careful planning. The goal of this capstone will be to answer the question: what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade? The findings will then guide a curriculum design that will inspire more flexible thinking and allow students to develop problem solving strategies that build on one another to allow for more meaningful and applicable cognitive understandings of the fundamentals of mathematics, specifically within the operation of multiplication. To begin, I will first take a look at the Constructivist Theory and how it can be applied to math to promote individualized learning and growth. Then I will review literature on the trouble with teaching only the standard algorithm with little room for variation or other strategies. Finally, I will review literature about the trajectory of strategies for solving
multiplication problems, the importance of developing number sense and flexible
problem solving strategies for multiplication problems, as well as using story problems to
promote active engagement of students. This literature review combined with
incorporating best practices of engaging teaching, will lead to the objective of this
capstone: the development of a curriculum guide that will allow students space and time
to develop strategies for internalizing strong number sense and problem solving
strategies, and in turn lead students to success as mathematicians.
CHAPTER TWO

Literature Review

Introduction

When designing a curriculum around student-centered problem solving strategies, one of the biggest challenges is figuring out how to give students the space and time they need to develop their own understanding. The teacher, acting as a guide and not employing direct instruction to reach the whole class, needs to have a good understanding of where the students are developmentally and be able to access their knowledge to help them grow in their understanding through experiential learning. The goal of this literature review is to answer the question of what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade.

This literature review will begin by summarizing Constructivist and Constructivist Learning Theory and how they pertain to student learning within mathematics. This will be followed by an overview of the trajectory of learning multiplication, the hierarchy of student thinking and problem solving strategies. Then the role of the teacher, as well as the importance of task selection and mindful guidance, will
be presented. Followed by literature that discusses the trouble presented when the algorithm is taught in isolation. Lastly, opportunities that rise from problem solving and how a student-centered approach can lead to success in developing a deeper understanding of multiplication.

**Constructivism**

First, consider Constructivist Learning Theory, perhaps the foundation of student-centered instructional practices, and one of its biggest proponents, Jean Piaget. In the years since Piaget’s theories have surfaced, researchers have examined mathematics instruction through a Constructivist lens and some have developed ideas about how this can support student growth. This involves the student’s direct involvement and development of their own problem solving strategies, as well as the teacher’s role as a facilitator of discussions towards developing understanding of big ideas rather than instructor. The following literature review will focus the educator’s lens on student centered math instruction. What happens when students are given opportunities to explore mathematics, rather than taught a procedure. The goal of this literature review is to examine a student-centered Constructivist approach to math instruction that will be the foundation for the development of a curriculum that will support the establishment of effective problem solving strategies and provide insight to whether or not it can lead to more flexible logico-mathematical thinking.

Piaget is well known for his ideas about the developmental stages that children go through as they progress through their cognitive and experiential understanding of the world around them (Piaget, 1964). Constructivism as a teaching model is built on the idea
that students bring their own prior knowledge to the classroom and their experiences within the classroom -- via direct instruction, collaboration or active participation -- add to that knowledge as they work through problems and are introduced to new ideas. However, Piaget argues that students must be active participants in the process in order to successfully internalize the material in a meaningful way. “To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed” (Piaget, 1964).

Vygotsky’s work centered on the idea that knowledge is constructed through social interactions. Vygotsky (1935, republished 2011) posits in his argument for the Zone of Proximal Development “that what is indicative of the child’s intellectual development is not only what he can do himself, but probably more so what he can do with the help of others” (p. 203). To clarify the ZPD Vygotsky writes;

The ZPD of the child is the distance between the level of his actual development, determined with the help of independently solved tasks, and the level of possible development, defined with the help of tasks solved by the child under the guidance of adults or in cooperation with more intelligent peers (p. 204).

This theory is foundational for the ideas expressed in this project. From the teacher selected tasks and guidance, to the discussions focused on student work and hierarchy of sophistication of strategies and problem solving. These will be unpacked more thoroughly in chapters three and four.
Blending Piaget’s ideas surrounding development of mathematical and logical thinking, or logico-mathematical thinking, and Vygotsky’s theory of ZPD and learning theories it makes sense that pre-primary and primary students are often given manipulatives, such as Unifix® cubes, blocks or beans, to use for developing beginning number sense and in order to practice adding and subtracting. According to Piaget’s theory of developmental stages, younger students are believed to be in the pre-operational or concrete operational stages (Piaget, 1964). Children within these stages aren’t as able to visualize or think abstractly therefore the use of manipulatives or active use of physical objects for the students to construct an understanding of the underlying cognitive and logical structures can be integral to their developing understanding. Add to that Vygotsky’s ZPD and introduce teacher selected tasks as well as discussion about strategy and the sharing of ideas and students are more able to unpack the learning.

What also comes in to play here is the flexibility of children’s thinking and how they understand a problem, and how their strategies may differ. In the Wells & Coffey 2005 article titled Are they wrong: Or did they just answer a different question?, the authors identify that teachers can promote active reflection of wrong answers to deepen mathematical understanding stating, “A teacher who takes the time to determine what question a child has answered is more likely to ascertain what mathematics the child truly understands, then build on it” (p. 204). In addition, they suggest moving away from “right or wrong” methods and instead focus on building confidence in problem-solving to foster genuine understanding by allowing kids to explain their answers and reasoning. This idea has been loudly echoed in the decade following their article. In 2015, Education Weekly
published an article highlighting a study that researchers Melissa M. Soto and Rebecca Ambrose were in the midst of employing focusing on the importance and instant benefit of formative assessment, specifically using technology. But the foundation is the same, with or without fancy software; the ability to allow teachers to “go beyond determining whether students correctly solved the problem, to understand why students solved the problem the way they did” (Herald, 2015; Soto & Ambrose, 2016). The idea is that students are able to record the steps of their process in writing (or drawings) and verbalize their strategy. The researchers concluded that this formative assessment has “the potential to transform the learning environment by allowing teachers to gain more in-sight into their students’ mathematical thinking” (Herald, 2015, p.12)

**General Trajectory of Learning Multiplication**

Each student will learn a little differently and at a different pace, but some work has been done looking at the general trajectory of students’ strategies and understanding of multiplication as an operation. This is relevant and important to consider when planning tasks and discussions and the order in which to present student work to push learning in a more effective and meaningful direction.

Carpenter et al. 2015, Baek, 2006, and Brickwedde, 2012 propose the following trajectory (see Figure 2.1):
These authors posit presenting students with word problems that feature equal groups is an entry point to multiplication. See Figure 2.2. The students are tasked with making units of units, using the multiplier to create the number of groups and the multiplicand to fill in the number of objects in each group\(^1\).

\(^1\) In a standard multiplication equation, the social convention used in this capstone places the multiplier is first, followed by the multiplicand and lastly the product. For example, 3 \(\times\) 5 = 15 with 3 as the multiplier, or number of groups, 5 as the multiplicand, or number in each group and 15 as the total number of objects in all the groups.
Repeated addition would be the next step for students. See Figure 2.3. Rather than counting each object as they put them into groups, as the tally marks show above, the students would either mark the five in each group with the numerical symbol 5, or they would simply add up three fives.

![Figure 2.3 Repeated Addition](image)

Skip counting is the next step, not having to draw a picture or add it up on paper but simply counting by a number 2, 4, 6, 8, 10, 12… or 3, 6, 9, 12, 15… often times students will use their fingers to keep track of the number they’re on.

![Figure 2.4 Skip Counting](image)
The last in the progression is Multiplicative Strategies. This includes decomposition and the distributive property. In Figure 2.5, the students are able to break down the factors of the numbers in the equation and work with them as parts of wholes. Thus being able to rely on learned facts to complete more challenging problems.

**Figure 2.5 Multiplicative Strategies**

![Equation](image)

Sherin and Fuson (2005) researched the taxonomy of single digit multiplication and the processes students use to problem solve. They break down the strategies into four areas; 1) semantic types, or word problems 2) intuitive models, or repeated addition 3) solution procedures, or computational and 4) models of retrieval, or fact recall. Within those strategies they propose the following progression (see Figure 2.6).

Student problem solving generally begins with counting everything (Carpenter et al. 2015; Baek, 2006). In this stage, “Count All”, the student needs to count each unit with the group, and keep track either in a drawing or on their fingers. After that students begin “Additive Calculation”, in this stage the student is able to add groups, but still represents each group with either a number or a math drawing. Students then move into
“Count By”, in this stage the student is able to skip count with the assist of drawings or keeping track on fingers. Next students progress into the “Pattern Based” stage. By now, students know some of the rules about multiplying by zero, one, five and ten. Some recognize that when multiplying by nine they can multiply by ten and then remove one from each group. Lastly students will be in the Learned Products stage, also called derived facts (Carpenter et al, 2015). This is the stage when the student can simply recall the correct product without showing their work.

<table>
<thead>
<tr>
<th>Figure 2.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count All</td>
</tr>
<tr>
<td>1) Drawing situational</td>
</tr>
<tr>
<td>2) Math drawing</td>
</tr>
<tr>
<td>3) Rhythmic fingers</td>
</tr>
</tbody>
</table>
Additive Calculation
1) Repeated addition
2) Collapse groups & add

Count By
1) Count by with drawing
2) Count by with written groups
3) Count by with fingers
As students progress through these stages of problem solving their strategies become more complex and they are able to come upon some of the conjectures and big ideas of multiplication through their own practice of working with multipliers, multiplicands and products (Carpenter et al., 2015).
The Role of the Teacher

“The teacher plays an important role in the creation of a good learning environment, which encourages exploration, communication and reasoning,” (Yang, 1996). The teacher’s role is to aide each student in their understanding with more challenging material and push the big ideas that lie in their Zone of Proximal Development or ZPD. Vygotsky’s theory of ZPD is that students’ abilities vary and it is less dependent on an individual’s maturational age and more on his or her cognitive age (Vygotsky, 1935, republished Vygotsky & Kozulin, 2011). The ability of a student to work at an independent level and make meaningful connections and internalize what they learn varies, and the teacher, or guide, scaffolds the material and the amount of help and direction given to lead the student to further understanding that would not be accessible, or would lead to frustration, if left to discover independently.

In more traditional teacher roles, the teacher’s task is to explicitly and directly instruct the students on specific algorithms or conduct rote memorization drills. There is usually a direct path to the end goal of mastery of operation and memorization of facts. What this combination of Piaget’s developmental theory and Vygotsky’s ZPD means for a Constructivist approach to math instruction is that it is the role of the teacher to allow space for students to develop their own understanding, what fits their cognitive age and understanding that they bring to the classroom and then scaffold the learning environment to allow them to grow in their understanding through natural experiential hands on learning opportunities (Piaget, 1964; Vygotsky & Kozulin, 2011).
In the years since Piaget and Vygotsky, there have been many researchers and educators who have tried on this idea of student centered learning and studied how the landscape of mathematical instruction changes when the development of logico-mathematical thinking becomes the primary focus. Some of those researchers are Thomas Carpenter, lead author of Children’s Mathematics: Cognitively Guided Instruction (1999, 2015) and James Hiebert, lead author of Making Sense: Teaching and Learning Mathematics with Understanding (1997). Both have promoted teaching for understanding and creating a student-centered experience for developing mathematical minds.

Hiebert et al. (1997) includes the idea of meeting students where their understanding is when he writes about figuring out how “information about students’ thinking indicates how students might enter the situation and how they might leave. This is valuable for selecting tasks that connect with where students are and that pull them in appropriate directions” (p. 35).

Once the teacher has a gauge for where the students are at in their own understanding, it is important to cognitively guide them to reasonable and logical comprehension of the big ideas as well as challenge them to further push their understanding and their ability to generalize and apply it in given situations. The idea is to create a learning environment that leads students to important residues (Hiebert et al., 1997). These residues are what are left behind from the student’s interaction with the big idea - what we want them to understand and build on. The idea then is that teachers select tasks that lend themselves to this vision of residue.
These carefully selected tasks encourage reflection and communication about the students’ ideas and strategies (Hiebert et al., 1997). Hiebert also includes in the teacher’s role as a guide the necessity of accessing a student’s prior knowledge and pushing him or her towards further exploration and deeper understanding of the material, often referred to as scaffolding. Providing direction for the activities or tasks and guiding the development of classroom culture while promoting autonomy is at the heart of the teacher’s role. The amount of teacher assistance and the range of difficulty of the material vary from student to student. What is important is that the students are given the opportunity to work independently with material and ideas that are accessible to them at their developmental level and from there they can develop better understanding of the material. An example of this taken from Hiebert’s work in the book *Making Sense* (1997) when applied to teaching multiplication might look like the following (see Figure 2.7).
Use of story problems that describe multiple groups:

“If Jermaine has 35 gumballs and 5 baggies, he wants to put the same number of gumballs in each baggie. How many gumballs can he put in each baggie.”

Interaction, conversations about what the problem means and differences between problems and strategies.

Introduce the “x” symbol for multiplication operation. Write number sentences.

Give the students problems that involve extending their thinking:

“If you have three different kinds of meat and two different kinds of cheese, how many different kinds of sandwiches can you make putting one kind of meat and one kind of cheese on each one?”

Continue to connect these ideas within interdisciplinary contexts throughout the rest of the year. (for example, in Science, using seedlings and making predictions about their growth if constant over time).
The teacher who employed this flow of tasks and corresponding conversations had a goal, and it was not to expose the students to a specific algorithm and focus on memorizing multiplication facts but to “allow the students to construct at least one solution method that they understood, and to develop the sense that multiplication encompasses a variety of problem situations.” (Hiebert et al., 1997, p.33). Also of note in this process is that the teacher chooses not to introduce the “x” symbol for the operation until after the students have worked through strategies to solve this type of problem. This is also supported by Jung, Kloosterman & McMullen (2007) when they posited “over time, children naturally begin to write number sentences to solve their problems, but teachers do not introduce formal ways of writing mathematics and of solving problems until children are comfortable with their own strategies” (p. 51).

While taking a step back from presenting information through direct instruction and being more observant and purposeful of planning tasks and discussion based on the students’ presenting understanding may not fit the traditional idea of the teacher’s role, it is integral in guiding students to expand and explain their thinking and develop sound mathematical understanding.

The Trouble With The Algorithm

Most mathematics curriculums include teaching students the standard algorithms. In the case of subtraction, in order to complete the problem the students must “carry” or “borrow.” Some may be familiar with these terms, because they were taught them, but they most likely did not discover them and uncover their meaning through personal experience with the material. Ewing and Kamii (1996) wrote that this approach, of
teaching from this associationist-behavioristic principles approach, is harmful to children’s development of logico-mathematical knowledge. They gave two reasons clarifying the harm caused. The first, as mentioned, are the rules of “carrying” and “borrowing” explaining that by teaching these rules teachers stifle children’s creativity requiring that they only work from right to left, or from ones to tens to hundreds to thousands and so on. They posit,

when children are free to do their own thinking, however, they invariably proceed in the opposite direction, from left to right. To add 38 + 16, for example, they typically do 30 + 10 = 40, 8 + 6 = 14, and 40 + 14 = 54. To subtract 18 from 32, they often say, "30 - 10 = 20. I can take only 2 from 2; so I have to take 6 more away from 20; so the answer is 14" (Ewing & Kamii, 1996, p. 260).

Along with limiting the directionality of their problem solving, the standard algorithms do not give precedence to place value and stand in the way of developing sound number sense. Take Ewing and Kamii’s (1996) example of a multiplication problem

...children’s typical way of doing 5 x 234, for example, is: 5 x 200 = 1,000, 5 x 30 = 150, 5 x 4 = 20, and 1,000 + 150 + 20 = 1170...while solving the preceding multiplication problem, for example, children who are taught algorithms say:

Five times four is twenty, put down the zero, and carry the two. Five times three is fifteen, plus two is seventeen, put down the seven, and carry the one. Five times two is ten, and so on. Treating every digit as ones is efficient for adults, who already know that the 2 in 234 is 200. For primary-age children, who have a
tendency to think that the 2 in 234 means two, however, algorithms reinforce their "errors." (Ewing & Kamii, 1996, p. 260)

The procedures taught focus less on place value and number sense and more on following the steps. Errors can then occur, but students with less developed number sense do not recognize it. They trust the algorithm. They may struggle with the steps that they do not fully understand, and then get to an answer that is incorrect, but they are not aware of it (Baek, 2006; Ewing & Kamii, 1996).

In the case of multiplication, and referring to the aforementioned sequence presented, the operation symbol “x” isn’t even introduced to the students until they have had ample personal experience solving problems direct modeling with manipulatives or drawing pictures. Jumping right to the standard multiplication algorithm, or teaching students to count the zeros can interrupt or deepen misunderstandings about the base ten number system. Carpenter states:

Although it can seem efficient in the short run, these procedures do not help children develop an understanding of our base ten number system. Children live in a world where they need to understand both very large and very small numbers. The strategies that children naturally develop to reason about Multiplication and Measurement Division problems with groups of ten, one hundred, one thousand, and so on help them develop this understanding (2015, p.91)

When students are taught the standard algorithms as procedures such as addition, subtraction, multi-digit multiplication or long division, they may be able to follow the steps and arrive at the answer, but conceptually they may not understand the mathematics
behind it all (Baek, 2006; Carpenter et al., 2015). Kilpatrick (2001) argues that, “When students practice procedures they do not understand there is a danger they will practice incorrect procedures, thereby making it more difficult to learn correct ones” (p.122). Students can misuse or confuse the standard algorithms and make mistakes within the procedure, then in turn they arrive at the wrong answer, but because they were following the procedure they do not realize their mistake and then have a false confidence in their ability to successfully solve math problems (Carpenter et al., 2015). This could look like a student attempting to solve a multiplication problem like 35x6 following the standard algorithm:

\[
\begin{align*}
35 \times 6 & \rightarrow 5 \times 6 = 30 \rightarrow 6 \times 3 = 18 \rightarrow 30 + 18 = 48 \\
\text{Rather than:} \\
35 \times 6 & \rightarrow 5 \times 6 = 30 \rightarrow 6 \times 30 = 180 \rightarrow 180 + 30 = 210
\end{align*}
\]

Teachers may not always take the time to determine the students’ conceptual understanding of the math when they are able to get a correct answer. Alternatively, when a student arrives at an incorrect answer the teacher may simply repeat the procedural rules rather than breaking down the mathematics of the operation. Reiterating the importance of Wells & Coffey (2005) here, “A teacher who takes the time to determine what question a child has answered is more likely to ascertain what mathematics the child truly understands, then build on it” (p. 204).

In the above equation 35 x 6 = 210, a teacher who looks closely at the work might realize that the student has their math facts down, they knew 5 x 6 = 30 and 6 x 3 = 18, but the neglected to see that it was truly 30 not 3. There is then opportunity to discuss
with them their strategy and guide them to direct model or attempt to decompose the number so that they see the two different values that make up 35. These conversations can lead to a deeper understanding of where the student’s understanding lays, and may highlight the need for more flexible problem solving strategies (Carpenter et al., 2015; Wells & Coffey, 2005).

**Opportunities From Problem Solving**

The biggest difference between the traditional teacher-directed approach and a Constructivist student-centered approach to mathematics instruction is the movement from teaching students’ procedures to allowing them to develop conceptual understanding of the mathematics behind the operations they perform through individualized learning opportunities. Giving students opportunities for developing problem solving strategies can give insight into their conceptual understanding and at the same time moves away from the idea of one “right” way and accepts multiple solutions to the same problem (Jacobs & Phillips, 2010; Wells & Coffey, 2005). The importance of formative assessments in math instruction was echoed by Kling and Bay-Williams (2014) who also argued that timed memorization tests do not paint an accurate picture of mathematical ability. Rather, an array of assessments including interviews, math games and writing activities that access thinking and strategies from multiple avenues show the level of mastery (Jacobs & Phillips, 2010; Wells & Coffey, 2005; Kling & Bay-Williams, 2014).

The role of the teacher is to facilitate this learning. Ladson-Billings (2000) states the teacher’s role within a Cognitively Guided Instruction construct is that teachers still
have to plan, but must leave room for student’s thinking, and allow that to guide lessons. They are to scaffold class discussions around the student’s own work and individual strategies, to clear up misconceptions, and to push students along, all the while allowing room for the student’s thinking, (Jacobs, 2010; Ladson-Billings, 2000; Carpenter et al., 2015) and promote active engagement in problem solving. Problem solving can and should involve using manipulatives, drawings or tallies to work out a problem involving two numbers operating on one another (For example, 4+□=7) but another characteristic of Cognitively Guided Instruction or CGI (Carpenter et al., 2015) is the use of story problems as a tool to understand the student’s thinking strategy for solving a given problem. In research conducted by Turner, Celedón-Pattichis, Marshall & Tennison (2009), they set out to identify specific instructional practices that teachers use to help students solve problems and communicate their mathematical thinking. They found that when teachers use stories or examples for presenting math problems that stem from student experiences, these stories allow students access to explanations. This gives the students an opportunity to have an active role in the process of completing the operation, which lends itself to conceptual understanding and cognitive growth (Piaget, 1964). For example, they can hear or read a problem like: Trevor had 3 boxes of sidewalk chalk. Each box had 8 pieces of chalk. How many pieces of chalk did Trevor have? They are able to visualize and illustrate the story thereby solving the problem. In this way, they are arriving at an answer that is correct and makes sense of the operation (Turner et al., 2009). Carpenter et al. (2015) wrote about this very thing stating, “In getting started, posing a problem that can easily be directly modeled offers the greatest possibility that
students will be successful.” (p.135) The students are then responsible for explaining their thinking, and giving them story problems allow them to explain their thinking in the context of the story. To aid in conversation, another tool to understanding students’ thinking, Carpenter recommends pairing up students to share with each other prior to a whole group share this allows them to verbalize their thinking and practice the language first. Moreover, CGI is about attending to student thinking, facilitating conversations and guiding them towards big ideas and conjectures by interpreting the students’ thinking, asking thoughtful questions and adjusting instruction accordingly to meet the students where they are at developmentally in their understanding (Carpenter et al., 2015).

One may think that students must understand the procedure in order to perform the operation and solve the problem. Using the example of Trevor with his sidewalk chalk, it is possible to read the problem and extract the equation $8 \times 3 = \Box$, but what Turner et al. (2009) found was that teachers didn’t wait for students to have all basic skills; instead, they used story problems to help students develop their understanding of the operations. In fact, when students are given the space and time to develop their own strategies and then share those with each other, maybe first in small group or with a partner, and then in large group the teacher can illustrate multiple representations that students present which can create more opportunities for understanding (Turner et al. 2009; Carpenter et al., 2015).

Another result of this practice is that the students become adept at discovering the underlying mathematical relationships with the story problems. Carpenter et al. (2015) says that,
Students learn that whether the problem is about monkeys or squids or a student in the class or about a stranger does not make a difference in the strategy they choose. Rather students learn to look for the mathematical relationships that are a part of the strategy and use them to get started on a solution (p.139).

Similarly, and in support of the growth that can come from the application of these invented strategies, Jung, Kloosterman, & McMullen (2007) state that, “When children's intuitions are respected and valued, and when they are encouraged to listen to other children explain how they answer questions, they naturally pick up more advanced ways of solving problems,” (p.55).

The conversations around the strategies play an integral part when guiding students to develop deeper understandings of the big ideas of mathematics. The teacher’s role with individual students will vary, but the teacher’s thoughtful questions and prompts can promote reflection and help give language to the strategies the student develops (Carpenter et al., 2015).

Maldonado et al. (2009) encourage the teacher to treat students as competent problem solvers and model the types of questions they should ask one another. Teacher should create opportunities, draw on multi-linguistic and non-linguistic resources, clarify and reenact ideas in the public space, as well as provide opportunities to solve related mathematical tasks. Moreover, when meeting with the large group and discussing multiple solutions and strategies the teacher can help illustrate the strategies by drawing on the board or showing student work. While asking the students to explain and stretch their own abilities in defining the concepts within the procedures they perform, the
teacher is nurturing connections, making the strategies visible and accessible to the rest of the class, again leading to more access to the material (Brickwedde, 2012; Jacobs & Ambrose, 2008). Teachers are also responsible for sequencing the presentation of students’ work. Trafton (1997) stresses importance of discussion among students and how this can lead them to different strategies. “Strategies are highlighted as they occur in children's presentations of their work. Thus, we do not "teach" particular strategies through lessons. When an important strategy emerges, we discuss it with the children in our seminars. We encourage them to apply it in other situations. We may not teach strategies, but we make certain that children learn them over time.” The idea is that while students explain their thinking and explore the work of others strategies, perhaps more efficient ones, will be revealed to them and they will be able to extend their thinking and deepen their understanding.

**Conclusion**

When students are taught the standard algorithms for mathematical operations and not given the freedom to explore and deeply develop their cognitive conceptual understanding, their logico-mathematical experience can be stunted. The literature shows that when given time and space to think and grow in their learning, students are very creative and capable when it comes to developing effective strategies. Piaget (1964) asserts:

This is what [logico-mathematical] experience is. It is an experience of the actions of the subject, and not an experience of objects themselves. It is an experience which is necessary before there can be operations. Once the operations
have been attained this experience is no longer needed and the coordinations of actions can take place by themselves in the form of deduction and construction for abstract, structures. (Piaget, 1964, p.180)

Mathematics should be presented to children as an opportunity to build strategies and stretch their minds. Their intuition and creativity should be appreciated, nurtured and celebrated. The teacher’s role is to carefully select tasks, curate conversations and allow students to explain their thinking and explore the concepts through interactions with the mathematical relationships and conversations with one another.

Chapter Three includes an outline for the development of a curriculum that supports this kind of learning environment. It pushes students away from trusting the standard algorithm to providing them with opportunities to experience and internalize mathematical operations and construct a fundamental understanding for mathematical concepts, laying the groundwork for our future mathematicians and answering the question, what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade?
CHAPTER THREE

Design Process

Introduction

The work of Piaget (1964), Carpenter et al. (1999, 2015), Brickwedde (2012), Kamii (2008), Ladson-Billings (2000), Bay-Williams (2010), among others have laid a foundation of understanding that children learn the concepts within mathematics best when given the freedom to explore and think for themselves, rather than be told how to do it. Children are extremely capable of unpacking and understanding complex mathematical ideas when given the chance to explore and attach meaning to them. Cornerstones of good teaching practices tell educators that we should include developing academic language, create shared experiences and use culturally relevant material and examples to engage students and allow them equal access to the material. What theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade?

The intention of this unit is to show that a practice grounded in Constructivist theory and student centered learning to explore mathematics would result in deeper and
more complex student understanding of multiplication and the conjectures within the operation. When students are given the opportunity to explore fundamentals of math they develop effective problem solving strategies that led to more flexible logico-mathematical thinking. This does not mean that when employing this approach the teacher sits back and watches the students work. Building curriculum and planning lessons is less about how to show the students the procedure for solving problems and more about how to guide the student’s thinking to get them to discover the big ideas through thoughtful discussion and purposeful tasks. The teacher’s thinking becomes: What language should I use? Which language should I avoid? What questions will I ask to push students’ thinking? How will I assess whether the students should be pushed or should they be allowed to stew a bit longer? All of this takes active listening and great flexibility.

Hiebert et al. (1997) talks about developing a “System of Instruction” in which teachers create lesson frames that honor the “learning trajectories” and set goals to get to big ideas, by thoughtfully choosing problems for the students to work on and facilitating the precipitating discussions. In practice, it is the students who determine the path they take to get there. As the teacher, I would be responsible to engage them in the mathematical ideas that help them to focus their understanding. If the students follow a misconception or their strategy does not follow a pattern or rule, the student’s thinking and strategies could have the potential to meander and remain unfocused. That being said, even with the strategic guidance, it is still very important that the students are involved in defining conjectures, that they make inferences about the big ideas in math
and help to create the language around the big ideas so that they can take ownership of the ideas and are able to move forward with a better understanding of them (Carpenter, 2002 and Ladson-Billings, 2000).

**Setting and Audience**

So what would this look like in a classroom where I implement this teacher developed curriculum designed for this project? A classroom in which this project’s curriculum could be practiced would look like a bunch of third graders hard at work solving real-world math problems! My experience is based on teaching in an urban public school with a culturally diverse population (Hispanic, African American, Native American and Caucasian) with roughly 50-60 percent of the class as English Language Learners and 85 percent of students receiving free-and-reduced lunch. This racially and culturally diverse group of students was using their creativity to construct the big ideas of mathematics through problem solving and discussions around their ideas and those of their classmates.

This curriculum is built with third grade in mind, but could easily be adapted for older or younger students. This can be accomplished by adjusting the complexity of, and language used within the story problems, and the pace at which the teacher moves students through their explorations. This curriculum frame would be presented during the core math instruction time for the entire class, meaning there would be a large variation in skills and background knowledge of multiplication and number sense. Addressing the diversity and language levels of the students would also influence the planning and instruction. Depending on the number of English Language Learners (ELL
students) this would necessitate scaffolding of academic language, providing word walls, sentence frames and opportunities to safely practice using the strategy discussion language.

While differentiation always presents a challenge for teachers, one of the benefits of using a constructivist, student-centered approach to teaching math is that the students could successfully learn from where they are developmentally, by meeting them where they are at and guiding and growing from there. Through discussions and sharing of ideas they can help each other see alternative methods of problem solving and push each other along in their understanding.

Best Practices and “System of Instruction” as Building Blocks for Curriculum Design

When creating lesson plans, units and curriculum, it is always a priority of mine to keep good teaching at the forefront and to keep students engaged and active participants in their learning environment. Most departments of education, including Minnesota’s, have a name for research driven, balanced, and rigorous curriculum that encourages student involvement and active participation- best practices. Teachers should always be learning themselves and reflecting on their practice (Carpenter & Franke, 1998). The design elements within the following curriculum included best practices, culturally relevant teaching, incorporated backwards design (Wiggins, 1950 & McTighe, 2006) and included key elements and tools for accessing student thinking based on the research of Cognitively Guided Instruction (Carpenter et al. 2015).
Drawing both from the literature as well as professional development experiences in which I have engaged, I have come to rely on certain instructional and assessment practices that inform my decision-making. The students are given a background knowledge assessment that allows me to access the students’ prior knowledge and experiences. This gives me a platform from which to launch the unit in a more meaningful way. By assessing the students’ background knowledge from the start their previous understanding becomes an integral part of the learning environment as well as influencing the learning trajectory. The use of formative assessments is necessary to give me a window into the students’ learning as we progress and a better understanding to continue adjusting the instruction and put appropriate scaffolds in place for student problem solving and discussions (Soto & Ambrose, 2016). By offering opportunities for background knowledge and formative assessments within each lesson I know what is grasped and what might need further review or simply if more time with the concepts is needed before the students are really able to internalize the big ideas or conjectures.

Maintaining cultural relevancy throughout the instruction, discussions and assessments is also important in the design of this curriculum. The aforementioned background knowledge and formative assessments are meant to give students many opportunities to show their understanding and progress. One of the essential principles of culturally relevant teaching as proposed by Banks et al. (2001) is that “teachers should use multiple culturally sensitive techniques to assess complex cognitive and social skills,” moreover Banks states:
Teachers should adopt a range of formative and summative assessment strategies that give students an opportunity to demonstrate mastery… Students learn and demonstrate their competencies in different ways…. Consequently, a variety of assessment procedures and outcomes that are compatible with different learning, performance, work, and presentation styles should be used to determine whether students are mastering the skills they need to function effectively in a multicultural society (p. 202).

It is also important to me to choose culturally relevant examples and make sure that all students are represented and feel a part of the classroom community. One way I will attempt to accomplish this is to use the student’s names as the subjects of the story problems, which automatically engages them! For the purposes of this curriculum, any student name used in a descriptive passage is fictitious. Such vignettes represent composite conversations based upon my previous experiences with students.

In a response to the ideas of Cognitively Guided Instruction, Gloria Ladson-Billings, whose work emulates culturally relevant teaching, states that CGI challenges the status quo, by nullifying the idea that math is for the elite few. She also claims that it also encourages students to think, which makes people nervous, because thinkers raise uncomfortable questions, leading to cognitive dissonance, which leads to learning (Ladson-Billings, 2000). Additionally, she posits that CGI will change the curriculum so that instruction will become less predictable.

Most of the research that has investigated the state of elementary mathematics in the U.S. indicates that our elementary mathematics curriculum is filled with rote
learning of low level arithmetic. The mathematics in the elementary curriculum is formulaic. Students are required to learn algorithms and rules for basic operations of addition, subtraction, multiplication, and division. Most students learn how to do those algorithms, follow those rules, and remember rote operations. However, most students do not learn what these operations mean. They do not learn how such operations might help them solve the kinds of problems that are important in their lives (p. 5).

In addition to culturally relevant teaching, it is important to offer instruction and practice that incorporates multiple modalities that honor the variety of multiple intelligences (Gardner, 1943) students may present as strengths or weaknesses. This will not only be considered for assessment, but also for instruction. Offering chances to work with new material from a variety of avenues and allowing students the opportunity for success within areas of their personal strength as well as in areas where they might have more needs. In this curriculum, this will look like many things. To reach the visual learners, number lines, pictures, manipulatives, word walls and demonstrating student work when discussing strategy will be used. To reach the aural learners, number talks, warm ups, and discussions focused on talking through strategies and encouraging active listening skills as well as opportunities to practice the academic language will be offered. Kinesthetic learners were given opportunity to use manipulatives, construct groupings using themselves and others move around through the work time to different groupings, and activities that included posting problems around the space.
Incorporating the idea of Backwards Design (Wiggins, 1950, & McTighe, 2006) and Cognitively Guided Instruction (Carpenter et al., 2015) practices guided the task selection and questioning used throughout the curriculum. By looking at the standards (see Table 1) and learning objectives, I will choose which problems to pose to the students. While this is representative of backwards design it also incorporates some of the tools of Cognitively Guided Instruction, the noticing, the unpacking, and active listening and discussions, the classroom discourse and routines of the students sharing their strategies.

Based on the students’ chosen strategies, I will then observe their work and decide whose work to project and process with the rest of the class. For example, on a day when the objective is to think about the distributive property I might post a story problem that features a two-digit by one-digit multiplication problem. Based on previous student work, I knew that problem solving strategies varied but would include; creating equal groups and counting by ones. Others would use repeated addition, while still others would break apart the numbers using the array model or simply by take the number apart by place value (Carpenter et al., 2015 and Baek, 2006). Knowing this I would able to ask students to share their work progressing through the levels of understanding. Thus, I use the discussion and student engagement to push them from less efficient strategies to more complex understanding of the distributive property. From a social constructivist lens, this discussion would be important because those students that didn’t yet see the numbers as parts of a whole, or that didn’t see the two-digit number as tens and ones, will be pushed
to feel the cognitive disequilibrium, a cornerstone of learning according to constructivism.

When what we experience differs from the expected or intended, disequilibrium results and our adaptive (learning) process is triggered. Reflection on successful adaptive operations (reflective abstraction) leads to new or modified concepts (Simon, 1995 p.115).

Conclusion

If this classroom, while using a Constructivist, student-centered approach to math were observed, the atmosphere would look busy, with cubes and drawings strewn about and the teacher meandering through students hard at work. Or maybe the observation would take place while students are engaged in a large group discussions where it seems each student has developed a different way of solving the same problem, and each has a unique contribution to the conversation. Maybe the observer will hear the teacher steering students whose own strategies seem tedious and inefficient towards a classmate who has discovered a “short cut” and then give them time and space to try it out. The student-driven exploration and discovery will be noticed by the teacher via planful discussions in small and large groups. The observer will notice the academic language that is gleaned from the spirited and engaging discussions resulting in the students learning how to describe their thinking, a skill that has the potential to extend far beyond this mathematics curriculum.
CHAPTER FOUR

Project Design

Introduction

The goal of this capstone is to determine what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade. The intention of this project design is to highlight a multiplication unit at the beginning of the second quarter of the year. In my own practice this unit lasts roughly six weeks in late fall of the third grade year. The class will cycle back to multiplication in late spring with reviews in the form of warm ups and word problems throughout the year. This project will present a lesson frame and offer a clear picture of what the teacher and students roles are with vignettes of potential discussions and student strategies all rooted in State and National Standards. This chapter will present where the students’ understanding based on research typically lies before the implementation of such a curriculum, what the teacher’s role includes, how student learning potentially progresses during, and what essential concepts have been nurtured after completing the unit.

Within this chapter I will present a daily lesson plan frame and walk through a day-in-the-life of this curriculum. I will also discuss what my role as the teacher includes
as far as preplanning, instructional choices, task selection, guiding student learning and discussions to help the students make sense of their strategies to move them forward. I draw upon some examples of student work typical of that found in the literature to exemplify how students move through the taxonomy of multiplication and give some examples of potential discussion that I anticipate might take place as students practice their understanding of the academic language and the range of strategies they may develop as we progress through the unit.

I close this chapter with a summary of the results and thoughts about where the students thinking and development may grow from where this unit leaves off and now they may continue to build a more solid foundational understanding of the operation multiplication. My thoughts about the importance of adopting a curriculum like this one and what hurdles exist in the widespread use of this constructivist approach will be presented in chapter five.

**State & National Standards**

According to the ‘system of instruction’ identified by Hiebert et al. (1997) and Simon (1995), having clear mathematical objectives are essential to any unit. The objectives provide the guidepost for any adjustments to instructional tasks based upon student understandings. State and common core standards serve as such guideposts in the unit design here.

The Minnesota State Standards that frame this unit include: mastering the use of multiple strategies including arrays, equal groups, repeated addition, using a number line and skip counting. They also incorporate the use of real-world problems that include the
language “how many in each group” that can be presented in word problems that promote accessible modeling strategies. The students will then move into more sophisticated strategies after having developed an understanding of the commutative and associative properties as well as partial products as they begin to work with larger two and three digit numbers.

The Common Core standards involve the equal groups and array strategies as well as word problems, all of the aforementioned may incorporate the use of drawings and direct modeling as well. In addition, they are to master the ability to recognize fact families and plug in for a missing factor or variable when given an equation. Students should become fluent with all products of two one-digit numbers, for example they should know from memory 2x8=16, 4x9=36, 7x6=42 etc. They should also have a firm understanding of the commutative and associative properties. These standards, both the state and Common Core are the anchor for this curriculum design, for the trajectory, the lesson frame, and the assessments administered, beginning with the pre-assessment. See the master grid later is the chapter for the full list of Minnesota State and Common Core Standards.

**Pre-assessment & Formative Assessment**

As was discussed in the previous chapters I believe it is very important to assess what your students already know before beginning to teach. Not only does this validate their background knowledge and bring to light any existing misconceptions, but also guides the teacher to launch the learning from a meaningful space. To that end, I would administer a pre-assessment to the students that is designed to reveal what they already
have had exposure to including multiple models for multiplication including arrays, skip counting on a number line, repeated addition, word problems featuring equal groups, and horizontal and vertical multiplication equations. The results of such a pre-assessment is intended to reveal what exposure to skip counting by twos, fives and tens, what automaticity with doubling numbers, and the capacity to add repeated numbers.

Pre-assessments would include the models of multiplication that are featured in the State and National standards, including arrays and number lines showing skip counting. Students would also be given word problems featuring equal groups to solve and to match to a given problem. They will also be presented two-digit by one-digit and a three-digit by one-digit multiplication problems in isolation. My past experiences with launching a unit on multiplication with third graders have resulted in an understanding of a typical range of prior knowledge. Students will likely have a limited understanding of multiplication. Curriculum presented to the second graders in this district does include working on equal sharing and equal grouping, introducing the concept of multiplication. While this exposure to thinking about equal groups helps in their ability to problem solve, my past experience teaching third graders shows that they have limited understanding of multiplication as an operation beyond x0, x1 and x2. While the operation may stump them, they may have experience with skip counting by twos, fives and tens, doubling numbers and they often have the ability to add repeated numbers. Some are able to recognize what to do when shown the “x” but most will likely attempt to add the numbers, often making errors and clearly showing deep misconceptions. There may be a few who have success solving the equations, but they may not recognize that
multiplication can be represented in multiple ways, or that there are different strategies that can be used to think about and work with the operation.

All of the questions presented in the pre-assessment would be rooted in the standards (see master unit grid later in the chapter) and would allow for evidence of growth when measured against how the students have progressed at the end of implementing this unit. See Figure 4.1 for examples of pre-assessment questions.

<table>
<thead>
<tr>
<th>Figure 4.1</th>
<th>Math Concept</th>
<th>Example Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Array as Multiplication Model</td>
<td>⋆ ⋆ ⋆ ⋆ ⋆ ⋆</td>
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<tr>
<td></td>
<td></td>
<td>⋆ ⋆ ⋆ ⋆ ⋆ ⋆</td>
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<td></td>
<td>⋆ ⋆ ⋆ ⋆ ⋆ ⋆</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4×3=12</td>
</tr>
<tr>
<td></td>
<td>Repeated Addition</td>
<td>4 + 4 + 4 + 4 + 4 + 4 = 24</td>
</tr>
<tr>
<td></td>
<td>Skip Counting on a Number Line</td>
<td><img src="image" alt="Number Line Example" /></td>
</tr>
<tr>
<td></td>
<td>Skip Counting</td>
<td>4×7=7...14...21...28</td>
</tr>
<tr>
<td></td>
<td>Word Problem Featuring Equal Groups</td>
<td>Huxley had 4 piles of cookies. He put 5 cookies in each pile. How many cookies did he have in all?</td>
</tr>
<tr>
<td></td>
<td>Operation in Isolation</td>
<td>14×8=112</td>
</tr>
<tr>
<td></td>
<td>Operation in Isolation</td>
<td>509×4=2,036</td>
</tr>
</tbody>
</table>
In addition to the background knowledge pre-assessment administered at the beginning of the unit, formative assessments should be administered throughout the implementation of this curriculum that will focus on testing the students’ progression and ability to answer multiplication problems featuring a variety of strategies.

The purpose of this data is to show the range of strategies students already have to solve the presented problems, such as derived facts, conjectures like zero times a number is always zero, or a number times one is equal to the other number, fact recall, or if they still have trouble multiplying low value single digit numbers in a way that is effective every time.

What can be expected based on my previous experience with students with similar background knowledge is that the results of the two assessments will most likely show that starting with the foundational understanding that multiplication is essentially repeated addition would launch the students in the right direction.

**Daily Lesson Frame**

Below you will find a daily lesson plan from early on in this curriculum. (See Figure 4.2) These activities are demonstrative of the routine that occurs within the daily math instruction time. The warm up activity is intended to be presented in a 25-minute
A chunk of time. In my experience I’ve presented this to students separate from their core math instruction. This has been on account of scheduling issues. As a result, this has benefited the students, giving them some processing time in between the warm up practice and discussion, and the more in-depth lesson and work time that follows later in the day. The rest of the lesson and student work time components are designed to be presented and practiced during a 65-minute math block later on in the same day. If a 90-minute block is available a teacher might determine it best to keep the instruction more consecutive and do the warm up followed immediately by the core lesson and work time.

<table>
<thead>
<tr>
<th>Figure 4.2</th>
<th>Warm Up</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity:</td>
<td>Teacher Role:</td>
<td>Student Role:</td>
</tr>
<tr>
<td>Word problems and discussion of strategies.</td>
<td>• Write 3-4 word problems featuring equal groups, students should be able to model with pictures or representational models</td>
<td>• Solve independently on a whiteboard, be prepared to share strategy with large group.</td>
</tr>
<tr>
<td></td>
<td>• Offer additional number sets to work from to differentiate for your students</td>
<td>• Practice active listening and responding by asking questions and using academic language</td>
</tr>
<tr>
<td></td>
<td>• Mindfully select students to share their strategies.</td>
<td></td>
</tr>
<tr>
<td>Large Group Lesson</td>
<td></td>
<td></td>
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<tr>
<td>--------------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td><strong>Activity:</strong></td>
<td><strong>Teacher Role:</strong></td>
<td><strong>Student Role:</strong></td>
</tr>
</tbody>
</table>
| Presentation of images and equations to guide thinking and steer strategies for independent work time. | • Task selection  
• Teach academic language of “factors” and “product”  
• Anticipate misconceptions  
• Provide visual aids to illustrate practice equations  
• Select pages in student workbook for practice | • Practice using product and factor in a sentence in turn and talk. |

<table>
<thead>
<tr>
<th>Student Work Time</th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>Activity:</strong></td>
<td><strong>Teacher Role:</strong></td>
<td><strong>Student Role:</strong></td>
</tr>
</tbody>
</table>
| Students will be working in their workbooks on a set of equations, and 2 word problems. They will have access to manipulatives (blocks) as well as whiteboards and markers. | • Purposeful partnering to help move students to more efficient problem solving strategies.  
• Monitoring to look for misconceptions or misuse of strategies.  
• Questioning to check for understanding and for explanation of strategy, to push them along or change direction. | • Practice comfortable strategy and attempt a new one while working on problem sets.  
• Share strategy with a partner and use new academic language within discussion. |
Within the warm-up and number talk the students are to be given word problems to solve. They each are asked to solve using their chosen strategy on a whiteboard and then I would select three or four students to share their work. I would use the document
camera to project their work and they would walk us through their strategy. They will be encouraged to use academic language, and I would prompt them as needed. Prompts would include asking what different components of their drawings represented if they didn’t offer that information up independently. My role in choosing whose work to show is to be based on the sophistication of their strategy. For example, I would choose someone whose strategy included using tally marks to count by ones to be presented before someone who counted by 2s or 5s. The idea is that as students progressed through their understanding of multiplication and gained more fluency they will recognize that there could be a more efficient way and try that out the next time around.

Figure 4.3 provides an example of student work as it can be projected on the interactive board for public sharing.
Here is an example of how a student might describing their strategy for the following equal group word problem:

Rick was making pancakes. He had four plates and he put four pancakes on each plate.

How many pancakes did he make in all?

Teacher: Alright Eloise can you please walk us through your strategy?

Eloise: First I drew four circles to represent the four plates. Then I put tallies on each plate until I got to 16.

Teacher: In your strategy, what do the tally marks represent?

Eloise: In my strategy the circles are the plates and the tally marks are the pancakes. So four plates with four pancakes on each plate means there are 16 pancakes in all.

Teacher: Could you come up with a number sentence to represent your work?

Eloise: Four times four equals sixteen.

The teacher’s goal for this interaction is to elicit student thinking, to have them verbalize their strategy for others. In this hypothetical group share Eloise described the steps she took to arrive at the answer. If the other students couldn’t follow the steps, it would be an opportunity for them to ask clarifying questions.

Eloise: In my strategy the circles are the plates and the tally marks are the pancakes. So four plates with four pancakes on each plate means there are 16 pancakes in all.

Lena: How come you made tallies? Can you just write 4, like the number?
Eloise: Yeah, so it would be 4... 8...12, 13, 14, 15, 16. Yes that works too, but I might get confused when the numbers get big.

Lena: It’s faster, but yeah, maybe you can start with the numbers and then go slower, like count by 2s instead of ones like your tallies, like 4... 8...12 +2 is 14 +2 is 16.

Eloise: I like counting by twos; I’ll try that next time.

One thing that continues to impress me is a student’s ability to make their thinking accessible to others. Sometimes they are able to explain a construct to a struggling student in a way I wouldn’t have thought of. This discussion is how they can push one another further along in the sophistication of their strategies. This incorporation of the discussions, and the opportunity to share thinking and socially built understanding stems from the Constructivist Learning Theory and are foundational to this curriculum project.

**Anticipated Movement Through Taxonomy Within the Large Group Lessons and Resulting Discussions**

In this section, I would like to address the noticings of themes and patterns that the students potentially may present as a result of discussions around their strategies. To do that, I will now discuss the decision-making and task selection to be made while the students are working through presented problems and subsequent strategies for solving. The trajectory of learning as presented in chapter two laid out how students tend to move from “Count All” including math drawing and rhythmic fingers, to “Additive Calculation” including repeated addition and collapse groups and add, then “Count By”
including count by with drawing, count by with written groups and count on with fingers to “Pattern Based” and finally “Learned Products” (Sherin & Fuson, 2005).

To begin the unit, I chose to introduce tasks that would encourage students to explore multiplication as repeated addition, one of the additive calculation strategies. The reason for this is that based on the background knowledge assessment I anticipate seeing that I would need to begin with a more tangible approach; with visuals and options for manipulatives to demonstrate the operation. The following two tasks are designed to engage students in a conversation about the relationship between two conceptual ideas of multiplication: multiplication as repeated addition of equal groups, and multiplication as an organized array. First present to the students the question, Is 6x4 the same as 4+4+4+4+4+4? Followed by presenting an array of 6x4 and asking if the picture describes the same equation. Posing those questions and guiding the students’ discussion contributes to building understanding of multiplication as repeated addition. For English Language Learners visual representation can be an invaluable aid, and what is best for some students can benefit all. I would start off by showing them many different examples of real life arrays, egg cartons, cookies on a tray, window panes, rows of plants in a garden, from there we would move to representational arrays, dots or stars in rows and columns. See Figures 4.4 and 4.5 below.
Array models presented an accessible way for students to picture multiplication as repeated addition, and it's warranted to say here that typical third graders are much more comfortable with addition as an operation at this point in the year than with multiplication.

As students moved through the taxonomy of multiplication the sophistication of their problem-solving strategies progressed. It was very common for students to next move into thinking more in equal groups, or math drawings within the count all and count by strategies (Carpenter et al. 2015; Baek, 2006; Sherin & Fuson, 2005). Students could
begin by counting all in order to arrive at the product. For example, when given the problem $5 \times 4 = ?$ they would draw out five circles and then put four tally marks in each circle, then they would count up the tally marks to solve the problem and get the answer of 20. Other students would be able to employ the count by strategy at this point and recognize that they could draw the five circles and write the number “4” in each one, then count by fours to arrive at the product. See Figure 4.6 for a side by side comparison of how these two strategies differ (Sherin & Fuson, 2005).

<table>
<thead>
<tr>
<th>Figure 4.6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Count All</strong></td>
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<tr>
<td><img src="image1.jpg" alt="Count All" /></td>
</tr>
</tbody>
</table>

This difference in efficiency is important to notice when monitoring and deciding the order of work to present to the large group when it is time for discussion of strategies. These differences will be the key to helping students notice when they might be able to stretch their thinking and move to a more efficient strategy. While monitoring the students as they work it is the teacher’s job to look at the strategies they choose and ask questions in an attempt to access their thinking and sometimes in an attempt to push their efficiency to a more sophisticated level. An example of how such a conversation might unfold is as follows:
Teacher: I see you’ve decided to use the equal groups strategy, Atticus. Can you tell me about your work?

Atticus: Yeah, I drew some boxes, well, I drew 6 boxes of crayons. Then I put three (tally marks) in each of the crayon boxes. Then I counted 3, 6, 9,12…13…14…15…16…17…18. 18 crayons in all.

Teacher: That’s great, I like how you chose to draw rectangles to represent the crayon boxes. I heard you count by 3s at first. I wonder what would happen if you tried writing the number 3 in each box?

The suggestion to try writing the number 3 rather than tally marks was building off of the student’s ability to count by threes. Because the student wasn’t able to count all the way to the product by threes they may not have independently arrived at the idea to use the number symbol, falling back on counting by ones using the tallies. However, because he was able to begin counting by threes he showed that he possessed the fundamental understanding of multiplication in this problem as six groups of three, so it was within his realm of understanding and therefore appropriate to push his strategy to the next level of counting by threes and using the number symbol instead.

The bulk of this chapter has been written to describe what a daily lesson could entail as far as the structure, activity and potential student responses and strategies. I am including a master unit grid that present the State and National standards, the unit goals, the objectives for the warm ups, the instructional tasks, small group and independent practice. It also includes a list of academic language goals including technical and content specific language. I have also included in the grid the goals of assessments to be
administered through the unit. The intent of this master grid is to guide the teacher planning their unit with the objectives and guidelines for task selection and targeted learning that fit to the fundamental ideas behind this curriculum project.

<table>
<thead>
<tr>
<th>Grade 3 – Multiplication Unit Master Grid</th>
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<tbody>
<tr>
<td><strong>Minnesota State Standards:</strong></td>
</tr>
<tr>
<td>3.1.2.3 Represent multiplication facts by using a variety of approaches, such as repeated addition, equal-sized groups, arrays, area models, equal jumps on a number line and skip counting. Represent division facts by using a variety of approaches, such as repeated subtraction, equal sharing and forming equal groups. Recognize the relationship between multiplication and division.</td>
</tr>
<tr>
<td>3.1.2.4 Solve real-world and mathematical problems involving multiplication and division, including both &quot;how many in each group&quot; and &quot;how many groups&quot; division problems.</td>
</tr>
<tr>
<td>3.1.2.5 Use strategies and algorithms based on knowledge of place value, equality and properties of addition and multiplication to multiply a two- or three-digit number by a one-digit number. Strategies may include mental strategies, partial products, the standard algorithm, and the commutative, associative, and distributive properties.</td>
</tr>
<tr>
<td>3.2.2.2 Use multiplication and division basic facts to represent a given problem situation using a number sentence. Use number sense and multiplication and division basic facts to find values for the unknowns that make number sentences true.</td>
</tr>
<tr>
<td><strong>Common Core Standards:</strong></td>
</tr>
<tr>
<td>3.OA.A.1 Interpret products of whole numbers, e.g., interpret $5 \times 7$ as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as $5 \times 7$.</td>
</tr>
<tr>
<td>3.OA.A.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.</td>
</tr>
<tr>
<td>3.OA.A.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations $8 \times ? = 48$, $5 = _\div 3$, $6 \times 6 = ?$.</td>
</tr>
<tr>
<td>3.OA.C.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.</td>
</tr>
<tr>
<td>3.OA.B.5 Apply properties of operations as strategies to multiply and divide.2 Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find $8 \times 7$ as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)</td>
</tr>
</tbody>
</table>
**Unit Goals:**
- Students will be developing ideas and understanding of equal groups and multiplication.
- Students will be able to solve real world story problems, illustrate their thinking via drawing or number sentences and explain how they came to their solution.
- Students will be able to engage in conversation with one another and show their thinking, as well as explain it using academic language.
- Students will be able to create a working definition for multiplication and describe the big ideas of the operation.

**Warm-ups:**
- Practice using familiar as well as new and unfamiliar problem solving strategies.
- Communicate ones’ thinking strategies and compare and contrast strategies with others.
- Ask questions of one another and stretch thinking

**Instructional Tasks:**
- Gain fluency and flexibility in using a variety of multiplication strategies and in a variety of problems.
- Decompose numbers using properties of operation (associative and commutative) and maintain equality of values in order to reconfigure those subunits into a more productive and usable combinations; Example: $12 \times 5 = (10 + 2) \times 5 = 50 + 10 = 60$
- Consistently describe numbers using the language of value rather than of digits.
- Create working constructs and rules for the operation as a group.

**Small Group and/or Independent Practice:**
- Practice new strategies and applying developed rules.
- Solidify trusted strategies and practice communicating the process to others.
- Solidify derived strategies for multiplication facts.

**Language** to be introduced, practiced and assessed within discussion and student Explanation of strategies:
- Language Functions: explain, demonstrate, describe, discuss, compare and contrast
- Technical Vocabulary: equal group, each, distribute, multiply, multiplication, times, factor, product, repeated addition, array, model
Assessment:

- Teacher will administer a pre-assessment to assess background and prior knowledge of equal groups and multiplication as an operation. Use of flexible formative assessments to gauge student understanding as you move throughout the unit. Monitoring student’s use and strategy and language to adapt instructional choices and task selection.
- Teacher will administer formative assessments to guide teaching and establish need for re-teaching and further review.

What to look for:

- Students work on whiteboards showing solutions and steps to get there
- Ability to verbalize their thinking and steps
- Progress from counting by ones, to higher numbers, using number symbols rather than tally marks, more advanced thinking and derived facts
- Use of academic language and proper values, not just naming digits
- Fluency and automaticity with strategies and/or derived facts
- Misconceptions or misuse of strategies

Conclusion

The goal of this capstone is to determine what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade. So far I have discussed the importance of pre-assessments and formative assessments within the unit, and to close the unit it is important to administer a summative assessment as a measure of student growth. In my own teaching I spiral through the third grade math standards throughout the year, so the students know that just because we’re closing a unit it doesn’t mean we won’t continue our work with the concepts. That being said, I advise administering a summative assessment. The importance of a summative assessment is twofold. It demonstrates how much the students have grown; what understanding they have gained and how the sophistication of their strategies has progressed. Through the formative and now the summative assessment you are able to see how the students have moved through
the taxonomy of strategies, whether from counting each tally mark to being able to use numbers to represent factors within a problem, or maybe they’ve progressed to learned products and are able to solve multi-digit problems using the commutative property effectively. A summative assessment will also show what misconceptions may still exist and what ideas and constructs need to be revisited throughout the rest of the academic year. Depending on the complexity of the pre-assessment you administer, it can be a fulfilling activity for the students to go back through their pre-assessment and “retake” it to see how much they have learned throughout the unit.

Throughout the unit formative assessments would have been administered and student progress should be noted and continuously advised to determine the pace of lessons when students needed more time to stew with ideas and strategies. By the time the summative assessment is administered, the teacher would have spent a lot of time reflecting on the noticings and trends the students had demonstrated through their work, discussions and formative assessments. Therefore there have been many data points for analysis of growth and the summative assessment should just be a cap on the unit, a bookend to feature their most sophisticated strategies and ability to use academic language to describe their thinking.

This concludes the project design portion of this capstone. In chapter five I will reflect on the experience of creating this project, doing the research, writing pages and pages about third graders thinking about multiplication and where I see this unit design going from here. I will also discuss some of the limitations that I foresee if students don’t do what you think they might. As well as the limitations within the practice, prevalent
instructional strategies and what I believe needs to happen in order for mathematics instruction to lead to more student success and deeper understanding of multiplication.
CHAPTER FIVE

Conclusions

The goal of this capstone was to determine *what theoretical basis and best practices should be used to guide instruction that develops fluency in conceptualizing and executing multiplication strategies over the arc of third grade*. In this concluding chapter I will reflect on the experience of creating this capstone, the research, the writing, the immense amount of thinking and time spent. I will also reflect on what limitations I foresee and what hurdles may exist in the widespread use of this constructivist approach. To aid in the latter I will be referring to a book written by J. Kilpatrick (2001) that posits there are five strands to “mathematical proficiency.” I will reflect on how this curriculum design would help students move towards proficiency in a meaningful way. Finally, I will reflect on where I’ll take this curriculum design from here.

Reflections

To begin I would like to say that this has been a very lengthy process stretching over the course of three years. I began writing this capstone the summer after student teaching in fourth grade and then working as a K-8 building reserve without a classroom of my own. I then began teaching third grade and am presently finishing up my third
year. I have been working on this paper, not only researching and writing but also reflecting on my experiences teaching math to my third graders over the past three years. In that same time I have also experienced what it is like to work in a results driven environment, one were “procedural fluency” (Kilpatrick, J., 2001) often trumps true understanding.

While I’ve been reading articles and encouraging my students to delve deeper and solve problems from a space of developing meaningful strategies, some of my colleagues have been singing rote memorization songs and handing out timed multiplication fact quizzes. I will talk more about this when I refer to the Kilpatrick chapter and the hurdles moving forward but this warrants note here as well because it has added to the reflective nature of writing one piece over such a length of time. Having started so long ago with the research and writing of this paper a lot has required revisiting and my own thoughts to understand how I have adapted and changed based on newer research and my own experiences. This truly has been a living document as I have progressed through my first years of teaching.

In the book “Adding it Up: Helping Children Learn Mathematics” by J. Kilpatrick written in 2001, Kilpatrick addresses the five strands of “mathematical proficiency” those five strands include: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Kilpatrick states, “How learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving” (p.117). He argues that in order for students to show true mastery they must show it within each of the five strands. He argues that conceptual
understanding is far more important than procedural fluency, which is often taught through rote-memorization. He posits, “These are not the kinds of connections that best promote the acquisition of mathematical proficiency” (p.119). In addition, he stresses the importance of strategic competence and adaptive reasoning explaining that this means, respectively, that students have the ability to formulate mathematical problems, represent them and solve them, and that they can think logically about relationships among concepts and situations.

Kilpatrick presents results from the National Assessment of Education Progress from 1996. The findings show that students develop this mathematical proficiency unevenly and that they show the most proficiency in the procedural fluency strand. My own wonderings then are, why is this the case? Is it because procedural fluency is the easiest to teach? It requires direct instruction of facts and procedures that are to be memorized and repeated without much questioning or critical thought. It doesn’t require students to do much more than regurgitate information and it is far easier to assess whether or not they pass a test- they either got the right answer or they didn’t.

The final strand, productive disposition, is explained as seeing sense in mathematics, to perceive as both useful and worthwhile. In order for a student to achieve this they would need to spend quality time with the mathematics, developing strategies they trust and are able to meaningfully articulate. I do not believe that this could be accomplished through solely teaching procedural fluency or in a teacher-centered learning environment. I do believe that the research used to build the foundation of the curriculum presented in the capstone not only encourages teaching to all five of the
strands but that it embodies the message that in order to develop true mathematicians the students need to internalize their learning and develop meaningful conceptual understanding.

I have put a lot of effort into informing my own teaching to meet the needs of my students in an attempt to develop a more meaningful experience through learning mathematics. In professional development and team meetings I have voiced my opinions and the research that backs them, and the process of writing this paper has invigorated my involvement in these discussions and my attempt to further the betterment of instructional practices of my team and my school. This has not always been met with welcoming attitudes.

I would be remiss to omit mentioning that one of the hardest things about working with a curriculum like the one presented here is being able to sit back and watch students struggle and stew in thought without showing them a strategy or solution. But giving them the space and time to figure it out with some guidance and thoughtful questioning is what I believe will lead them to becoming problem solvers and true mathematicians.

**Limitations**

Many teachers are set in their ways, especially if they produce “results” in procedural fluency because this makes their teaching appear effective. One of the major hurdles I foresee in a curriculum like this one being adopted is that teachers would need to abandon the way they think about and plan their math lessons. I had a very different idea about teaching math until I was taught how to teach it by James Brickwedde at Hamline University. I imagined teacher-centered instruction followed by students
practicing the same strategies until they were able to reproduce the results without error-like how I was taught. Brickwedde encouraged us to leave that mentality at the door, as well as the idea that the standard algorithm was the one and only way to solve problems. Through practicing mental math strategies, explaining our thinking to peers, deconstructing the rules of math through pictures, word problems, number lines and balancing equations we changed the way we thought about math. The introduction to the research on student-centered math and Cognitively Guided Instruction (Carpenter et al., 2015) fueled my passion for teaching and learning math in a new way. This is how I became an advocate for a different way of teaching, but it truly took walking away from how I was taught. Is it reasonable to ask this of all teachers? Probably not, but it is worth having the conversations and showing them just what students are capable of when given the time and space to get creative and solve problems.

Another limitation presented with this capstone is that it has not yet been implemented so I do not have the benefit of actual student participation to help mold my instructional strategies or results of growth. I only have the research I’ve done and my experience up to this point. The vignettes of conversations in this capstone are fictional and based on my previous experience, therefore they are simply an educated guess and could very well not go as planned. However, the nature of teaching requires flexibility, especially with what we see from our students learning and progress. That is a limitation but also a component of this curriculum that is beneficial. As the teacher we need to be responsive and adaptive to the needs presented by our students.
Future Steps

Finally, I would like to address where I’ll be taking this from here. I am set to teach third grade again in the coming school year and it is my intention to adapt this curriculum to my future students. While this curriculum was based on multiplication, the ideas and processes imbedded within it can be adapted to teaching all mathematical subjects. Similar approaches can be applied to addition, subtraction, division, fractions, geometry, as well as working with number sense and place value. I intend to create a learning environment that is catered to student-centered learning throughout the academic year. I acknowledge that following the literature and research is imperative to incorporating current best practices and I recognize how I may need to adjust the crafting of my lessons in light of new findings. I will take this on with the best intentions as an educator and continuing student of mathematics.

I have read and though a lot about teaching math while working on this paper, and I have noticed how it impacts my professional development, and the way I take notice of my student’s learning. I intend to continue to do so, looking for new ideas and findings about effective instruction and student learning. I especially look forward to reading results of longitudinal studies of students who were immersed in student-centered math instruction throughout their educational careers to see how they have improved in their “mathematical proficiency” compared to the generations before them. I hope that my own research and projected ideas contribute to the advancement of mathematics instruction and lead to a future full of mathematicians who can solve problems and explain their thinking.
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ehost--live;; http://my.nctm.org/eresources/article_summary.asp?
URI=TCM2006--09-104a&from=B

Images used for examples of arrays:

Window Pane http://www.clipartkid.com/images/810/then-hung-it-securely-on-the-
siding-creating-a-faux-window-NPWPYG-clipart.jpg

Cookies
http://article.images.consumerreports.org/prod/content/dam/cro/news_articles/ap-
pliances/PamperedChef_cookie_sheet

Eggs https://s-media-
cacheak0.pinimg.com/736x/5e/7d/5c/5e7d5c5e984ca12c99be4fb1c322e240.jpg
APPENDIX A

Ten-Day Lesson Plan Framework

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-10 minutes</td>
<td>Launch&lt;br&gt;Get students thinking mathematically, generally using mental math strategies</td>
</tr>
<tr>
<td>25-30 minutes</td>
<td>Student Work Time&lt;br&gt;Teacher selected word problems</td>
</tr>
<tr>
<td>15-20 minutes</td>
<td>Discussion&lt;br&gt;Large group number talk students sharing their strategies</td>
</tr>
<tr>
<td>5-10 minutes</td>
<td>Wrap Up&lt;br&gt;Hone in on conjectures and big ideas, create or add to visual classroom display</td>
</tr>
<tr>
<td>Day</td>
<td>Objective</td>
</tr>
<tr>
<td>-----</td>
<td>---------------------------------------------------------------------------</td>
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| 1   | Students will become familiar with manipulatives and problem solve for 2 story problems. | • launch get to using 1-100 cards  
• main: Allow for play and exploration with manipulatives. Then display story problem, read through it make sure all understand language of problem. Allow time for students to work out problem with manipulatives and create strategy. Do this for 2 problems. Students will work independently and see what creative solutions they can come up with without direct instruction from the teacher.  
• Formative Assessment (FA): teacher observation of independent work. | • launch helps to develop number sense and strengthen students’ mental math skills.  
• allowing for exploration of manipulatives prior to working with them gives students the chance to play and become comfortable with the materials.  
• Giving students the space and time they need to work out problems on their own lends itself to better and deeper understanding of the operation through hands on experiential learning.  
• Paying close attention to how the students are working with manipulatives and how they are arriving at answers allows the teacher to determine their readiness to move on or what might need more time for development. |
| 2   | Students will be able to work through a story problem featuring multiplication. Students will be able to discuss their strategy with partner and large group. | • launch: double a number  
• main: Independent practice for 2 more story problems using manipulatives. Students will then share strategies with partner. Followed by large group discussion of strategies for the story problem. Teacher can base which students called on by the degree of complexity of their strategy, beginning with less complex strategies.  
• FA: Teacher observation of independent work and participation in discussion. | • launch develops number sense and strengthens mental math skills.  
• Give students more time to work out their strategies.  
• Once all students have had time then regroup to discuss. By purposefully calling on students in order of the progressiveness of their strategy the students can push one another to extend their thinking. The teacher’s role as facilitator is not only to choose students to explain their strategies but to help them develop the language needed to explain. |
| 3 | Students will be able to work through a story problem featuring multiplication. Students will be able to discuss their strategy with partner and large group. | • launch: difference between 
• main: give students white boards and whiteboard markers to begin drawing rather than using cubes for problem solving. Students will work through 3-4 problems independently. They are to do the last one on a half sheet of paper to be turned in to the teacher. Students will then discuss strategy with partner and then in large group. 
• FA: Half-sheet and teacher observation of independent work and participation in discussion. | • launch develops number sense and strengthens mental math skills. 
• by moving from manipulatives to drawings or tally marks the students are still using concrete operations but can use more flexibility, creativity and often begin to develop short cuts for solving. |
| 4 | Students will be able to work through a story problem featuring multiplication. Students will be able to discuss their strategy with partner and large group. | • launch: double/triple 
• main: Begin this portion with a large group discussion about place value and how students keep track of the tens and ones when problem solving. Talk about how would they do the same for hundreds? When students are working they should be talking to classmate about how they are keeping track of place value and using the appropriate names for numbers (twenty not two etc.) Students will again use whiteboards and draw out their solutions for problems rather than use manipulatives. 
• FA: Teacher observation of work on white boards as well as through asking questions while students are working. Half-sheet solving one problem. | • The discussion about place value is very important. When students are working with multi-digit numbers it is very common to hear students say something like “then you take the 2 and you have to make 4 groups...” when really they are talking about the 2 in 25 which is really 20. Having this conversation brings the importance of place value up and gives the students opportunity to practice talking out their strategies while keeping place value intact. (Brickwedde, 2012). |
| 5 | Students will be able to move through stations successfully developing a strategy for each station. | • launch: get to  
• main: Stations will consist of 8 different areas. (It is intended that students will get to 4 out of 8 on day 5)  
Working in groups of 3-4 the students will work their way through the stations- each will contain a challenging question and they will have to work together to solve and discuss to come up with one answer.  
• FA: teacher observation and group work turned in. | Stations allow for movement and a brain break as the students move from one to the other. The groups will work together sharing ideas and practicing their language as they attempt to find solutions for challenging questions together. |
| 6 | Students will be able to move through stations successfully developing a strategy for each station. | • launch: difference between  
• main: Same as Day 5, finish up unattended stations.  
Follow up with discussion about 2-3 of the more challenging questions based on the group’s experience.  
• FA: group work turned in and large group discussion. | Same as above. The discussion will be another chance for students to work through language and talk through their strategies, sharing ideas with one another. |
| 7 | Students will be able to identify the operation of story problem. | • launch: difference between  
• main: Students will be shown story problems that feature different operations (addition, subtraction, division and multiplication) and be tasked with determining the operation of the story problem in a large group setting and then students will talk through how they will solve using turn and talks and sharing out.  
Students will then be given a worksheet and whiteboards to work out mixed operations problems with a partner.  
• FA: participation in discussion and completion of worksheet. | • The operation featured in the story problems the students will be solving will have featured multiplication up to this point. By challenging them with story problems involving other operations it is another opportunity for them to extract the math action that is occurring and practice using their acquired language to express their strategies to others. |
| 8   | Students will be able to identify the operation of story problem. | • launch: get to  
• main: small group discussion then large group discussion, develop working definitions for conjectures.  
• FA: Student involvement in discussions and teacher observation of work in small group. | • This is the time that together the class develops conjectures, or working definitions, for the big ideas they are working with. It is a great way to access their conceptual understanding and put into words the operation and general rules about how it works. Students are then tasked with making sure that the conjecture is solid by testing it out with other problems, either given to them or ones they make up- proving to themselves and others that the big ideas hold for all numbers and situations. (Carpenter et al., 1999, 2015). |
| 9   | Students will be able to write story problems featuring multiplication for one another and share solving strategies successfully. | • launch: double/triple  
• main: Students will work independently to write a story problem that involves multiplication. They will then partner up and solve each other’s problems sharing with one another their solutions and strategies.  
• FA: Review story problems written by students. | • By having the students write the story problems you are able to tell whether or not they have developed the language around the operation, and whether or not they can successfully communicate the mathematics of the problem. Then when discussing the solution with their partner, they can practice with the language of their problem solving as well. |
| 10  | Students will be able to solve multiplication problems using effective strategy and explain how they came to the solution using academic language. | • launch: triple/quadruple  
• main: Students will work independently to solve story problems featuring multiplication and use language to write out their steps for solving.  
• Summative Assessment: worksheet featuring 2 story problems and 1 that includes language component detailing strategy. | • Give the students the opportunity to show off what they have discovered about multiplication. This summative assessment, combined with Day 9 Formative assessment should be very telling of the students’ conceptual understanding of multiplication. |