Comparison of the Sharpness of Tungsten Field Emission Tips from Traditional Electrical Characterization to Tip Geometries Imaged by Scanning Electron Microscopy

Emma C. Reeves

Follow this and additional works at: https://digitalcommons.hamline.edu/dhp

Recommended Citation
https://digitalcommons.hamline.edu/dhp/15
ABSTRACT. Using field emission data obtained from 11 etched Tungsten successful field emission tips of radius $\sim 20 - 100$nm, FN-type linear models were compared [4], [13], [16]. The emission tip radii were determined using an iteration method derived from the modified FN linear equation. In addition, Scanning Electron Microscopy images of 5 successful emitters were obtained. These images were fit to a circular model to estimate the actual radius and compared to empirically predicted radius values. A hyperbolic model was further fit to the images and the circle of similar apex curvature was derived. A method for calculating the electric field for these modelled geometries was suggested and a sample code has been provided for future research.
1. Background

1.1. Field Emission Theory. Cold field electron emission, commonly referred to as field emission, is the emission of electrons from a surface due to a negative applied voltage at that surface. The first account of field emission was published in 1897 by J.J. Thomson [11]. At this time, the classical quantum theory was not fully developed and the field emission phenomenon was only understood experimentally.

![Figure 1](image)

**Figure 1.** *During field emission in vacuum, emitted electrons incident on a phosphor screen cause photonic emission seen as blue light.*

With the rise of quantum theory came an understanding of microscopic phenomena. Fermi described the distribution of electrons, an example of fermions, in a material in 1926 [4]. In 1927, Sommerfield advanced Fermi’s result to specify the behavior of electrons in a metal [3]. Using these results, the field emission theory of electron tunnelling, now known as the Fowler-Nordheim theory, was published in 1928 by R.H. Fowler and L. Nordheim.

The structure of energy levels in a metal is shown in the figure (2). Neighboring energy levels, shown as yellow lines, are close and are approximated as continuous. Electrons in the conduction band are nonlocalized. Due to the lattice structure of a metal, the conduction and valence bands are separated by a band gap, a region with no allowed energy states. The band gap is significant compared to an electron’s energy so that only conduction electrons are expected to tunnel.
Electrons are fermions, meaning that no two electrons can occupy the same quantum state. There are a finite number of available states in each energy level, and the electrons will fill them from the lowest to the highest energy. At absolute zero, the highest filled energy level is called the Fermi energy, $\mu$ \cite{4}. For metals, the Fermi energy is typically several electron volts above the lowest allowed energy. The absolute temperature, $T$, at which $T = \frac{\mu}{k_B}$, where $k_B$ is Boltzmann’s constant, is much higher than room temperature, so the highest filled energy level at room temperature is also approximately the Fermi energy, $\mu$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{band_gap.png}
\caption{Image of the band gap in a solid conductor. \cite{9}}
\end{figure}
The potential energy well for an electron in the conduction band of a metal.

The height of the potential energy barrier for an electron at the Fermi energy level is called the work function, $\phi$, and is material-dependent. The work function can be measured using the photoelectric effect.

![Diagram of potential energy well for an electron in a metal.](image)

**Figure 3.** The potential energy well for an electron in the conduction band of a metal.

<table>
<thead>
<tr>
<th>METAL</th>
<th>WORKFUNCTION (eV)</th>
<th>METAL</th>
<th>WORKFUNCTION (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>4.3</td>
<td>Ru</td>
<td>4.7</td>
</tr>
<tr>
<td>Ti</td>
<td>4.33</td>
<td>Rh</td>
<td>4.98</td>
</tr>
<tr>
<td>V</td>
<td>4.3</td>
<td>Hf</td>
<td>3.9</td>
</tr>
<tr>
<td>Cr</td>
<td>4.5</td>
<td>Ta</td>
<td>4.25</td>
</tr>
<tr>
<td>Mn</td>
<td>4.1</td>
<td>W</td>
<td>4.55</td>
</tr>
<tr>
<td>Fe</td>
<td>4.7</td>
<td>Re</td>
<td>4.96</td>
</tr>
<tr>
<td>Co</td>
<td>5</td>
<td>Os</td>
<td>4.83</td>
</tr>
<tr>
<td>Ni</td>
<td>5.15</td>
<td>Ir</td>
<td>5.27</td>
</tr>
<tr>
<td>Nb</td>
<td>4.3</td>
<td>Au</td>
<td>5.1</td>
</tr>
<tr>
<td>Mo</td>
<td>4.6</td>
<td>TaN/TaSiN</td>
<td>3.9-4.3</td>
</tr>
</tbody>
</table>

**Figure 4.** Work functions determined for given metals. Tungsten, used in this experiment ($\phi = 4.5\text{eV}$), is highlighted. [10]

1.2. The Supply Function.
Electrons are fermions, meaning that no two electrons can exist in the same state. The probability that electron will exist in a state is given by the Fermi-Dirac distribution:

\[ P \equiv \frac{1}{1 + e^{(\frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) - \mu)/k_B T}} \]

where \( v_x, v_y \) and \( v_z \) are the components of electron velocity, \( k_B \) is Boltzmann’s constant, \( m \) is the electron mass, and \( T \) is the absolute temperature.

The traditional interpretation of the space a state occupies is given by the uncertainty principle in position-momentum phase space. That is,

\[ \delta x \delta y \delta z \delta p_x \delta p_y \delta p_z \geq \left( \frac{\hbar}{2} \right)^3 \]

where \( x, y, z \) are the cartesian spatial dimensions and \( p_x, p_y, p_z \) are the cartesian momentum dimensions, and \( \hbar = \frac{h}{2\pi} \) for Planck’s constant \( h \). In general, this phase space volume is approximated as the following, where \( \left( \frac{\hbar}{2} \right)^3 \) has been replaced by \( \hbar^3 \):

\[ \delta x \delta y \delta z \delta p_x \delta p_y \delta p_z \simeq \hbar^3 \]

To account for electron spin, the space is doubled:

\[ \delta x \delta y \delta z \delta p_x \delta p_y \delta p_z \simeq 2 \hbar^3 \]

Finally, \( p = mv \) is used to find the volume in velocity space:

\[ \delta x \delta y \delta z \delta v_x \delta v_y \delta v_z \simeq 2 \frac{\hbar^3}{m^3} \]

The number of electrons within an energy range, \( dv_x dv_y dv_z \), is the product of the phase volume and the Fermi-Dirac distribution.

\[ N(v_x, v_y, v_z) \delta v_x \delta v_y \delta v_z \simeq 2 \frac{\hbar^3}{m^3} \frac{1}{1 + e^{(\frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) - \mu)/k_B T}} \]

When integrated over the available energies in a material, this function is known as the supply function, \( N(T, W) \), where \( W \) is the total electron energy. The supply function is used in Fowler-Nordheim theory to describe the rate of electron emission from a metal.

1.3. The Exact Triangular Potential Energy Barrier.

At room temperature and without the aid of external forces, a conduction electron in a metal cannot escape the potential energy well in which it is confined. However, if the potential energy barrier can be narrowed so that its width is on the order of the electron’s uncertainty in position, then the electron may tunnel through and exist outside the barrier.

Such a thin barrier is achieved by creating a sharp metal tip and applying a high negative voltage to it. The resulting electric field acts as a force on an electron in the metal. As the electron tunnels into the potential energy barrier, the work done by the electric field linearly reduces the potential energy barrier. Therefore instead of a finite well, the potential energy barrier is triangular. This potential energy barrier is known as the Exact Triangular, or ET, barrier. In field emission, the applied voltages are high enough to allow electron tunnelling through this potential energy barrier.
Figure 5. $\phi$ is the potential energy barrier initial height, or work function. After application of a high voltage to the sharp metal tip, the resultant one-dimensional barrier is approximately triangular [3].

When the product of the supply function for electrons in a metal ($N(T,W)$ above) and the transmission coefficient for the ET barrier is integrated over the available electron energies, the result is the original Fowler-Nordheim, or FN, equation. This equation predicts the electron emission current density, $J$, for a known applied field, $F$.

\[
J = 6.2 \times 10^6 \frac{\mu^2}{(\mu + \phi)\phi^2} F^2 e^{-6.8 \times 10^7 \phi^2 / F}
\]

where $J$ is given in $\text{Amps/cm}^2$, $F$ is the electrostatic field strength in $\text{V/cm}$ and energies are given in $\text{eV}$.

Traditional field emission assumes that an emission tip is approximately some solid angle of a sphere. For an electron in this conducting sphere, the curved surface appears approximately planar. Therefore, this geometry satisfies the assumption that the potential energy barrier is one-dimensional. In particular, tunnelling will occur through the barrier in the outward surface-normal direction.
Figure 6. Cross-section of an approximately spherical emitter. Electric field due to a voltage applied to the conductor is denoted $E$ in this figure.

$$F \equiv E = \frac{q_{encl}}{4\pi\epsilon_0} \frac{1}{r^2} = \frac{V}{r}$$

where $r$ is the radius of the sphere. If the emission surface area, $A$, is constant, then the emission current density is given by

$$J = \frac{I}{A}$$

The Fowler-Nordheim equation can be linearized for experimental use by these identities for $F$ and $J$:

$$J = 6.2 \times 10^6 \frac{\mu^\frac{1}{2}}{(\mu + \phi)\phi^\frac{1}{2}} F^2 e^{-6.8 \times 10^7 \phi^\frac{3}{2} / F}$$

$$I = A \times 6.2 \times 10^6 \frac{\mu^\frac{1}{2}}{(\mu + \phi)\phi^\frac{1}{2}} \frac{V^2}{r^2} e^{-r \times 6.8 \times 10^7 \phi^\frac{3}{2} / V}$$

$$\ln(I/V^2) \propto 1/V$$

1.4. The Schottky-Nordheim Potential Energy Barrier. A more accurate potential energy barrier includes the image effect potential energy term. This image potential occurs due to the additional field induced by an electron near a conducting surface, such as the planar emission surface of a metal tip.
Figure 7. The charge induced on a conducting plane by an electron at point $(-x, y, z)$ can be represented by a point of opposite charge located at $(+x, y, z)$ according to the Uniqueness Theorem of electrostatics.

The resulting potential energy barrier is known as the Schottky-Nordheim, or SN, barrier. For energies measured from the work function, $\phi$: 
Figure 8. The Schottky-Nordheim (SN) potential barrier. All energies are measured from a point \( \phi \) above the Fermi energy, \( \zeta \). \(-W_a\) is the lowest allowed electron energy and \( x \) is the tunnelling direction [19].

\[
V(x) = -F ex - \frac{e^2}{4x}
\]

where \( e \) is the elementary charge and \( F e \) is the force on an electron due to the electric field. The image effect was recognized by Fowler and Nordheim in their original work [3] but was assumed to reduce the potential energy barrier by an insignificant amount. In 1956, Murphy and Good solved for a modified FN equation using the SN potential energy barrier [19]. This result improved upon the original equation, which had significantly underpredicted experimental emission current densities. The linearized modified FN equation can be given by

\[
\ln(I/V^2) = \ln(a) - 6.8 \times 10^7 \alpha kr\phi^2/V
\]

where \( a \), \( k \), and \( \alpha \) are constants and \( r \) is the radius of the emitter.

More recent studies have calculated the FN equation for different potential barriers, including hyperboloidal [20], [13]. The core theory of FN field emission remains consistent, however, and here some time is taken to review derivations for the original and modified FN equations.
2. The Fowler-Nordheim Theory

In order to derive a Fowler-Nordheim-type equation, it is necessary to find the transmission coefficient $D$ for a given potential barrier $V(x)$. The transmission coefficient is a measure of the magnitude of a particle’s wavefunction after tunnelling relative to the magnitude of the wavefunction incident on the barrier.

2.1. The Exact Triangular Barrier.

2.1.1. Gomer Triangular WKB Approximation.

Gomer published a thorough literature on basic field emission theory for primarily experimental use in 1961 [4]. His solution for $D$ for the ET barrier relies on the WKB approximation, a well-known quantum mechanical approximation for finding wavefunction solutions to the time-independent Schrödinger equation.

The WKB approximation states that for a wavefunction with magnitude $A$ incident on a potential energy barrier that is nonconstant in space, the wavefunction solution will decay exponentially inside the barrier. The approximation requires that $\frac{d^2 A}{dx^2} \ll (\frac{\phi}{dx})^2$ and $\frac{d^2 A}{dx^2} \ll (V(x) - E)$, where $\phi$ is the wavefunction phase, $E$ the total energy, and $(V(x) - E)^\frac{1}{2} \equiv p(x)$ is known as the effective potential function.

As shown, the wavefunction is required to be continuous and smooth at the potential well boundaries. The transmission coefficient, $\frac{|F|^2}{|A|^2}$ for the wavefunction with transmitted amplitude $F$ and incident amplitude $A$, is therefore proportional to the exponential decay function inside the barrier. The prefactor $f(E,V)$ given by the WKB approximation is dependent on the particular values of $|F|$ and $|A|$ and is typically on the order of unity. We note that although the complete WKB approximate wavefunction solution is a linear combination of decaying and growing exponentials, it is typical to assume that the decaying solution dominates.

![Figure 9. $D = \frac{|F|^2}{|A|^2} = f(E,V)e^{-\frac{2}{\hbar} \int_0^L |p(x)|dx}$](image)

Figure 9. $D = \frac{|F|^2}{|A|^2} = f(E,V)e^{-\frac{2}{\hbar} \int_0^L |p(x)|dx}$.  


After substitution of $p(x)$:

$$D(E, V) = f(E, V)e^{-2(\frac{2m}{\hbar^2})^{\frac{1}{2}} \int_0^l (V(x)-E)^{\frac{1}{2}}dx},$$

(15)

Here, we will introduce $E_x$. Energy is a scalar quantity and cannot have components, but it is common nomenclature in field emission to use $E_x$ as the surface-normal "component", or energy associated with outward surface-normal motion $v_x$. $V(x)$ for the ET barrier and an arbitrary electron energy $\mu - E_x \equiv E$ is substituted to find the solution:

$$D(E_x, V) = f(E_x, V)e^{-2(\frac{2m}{\hbar^2})^{\frac{1}{2}} \int_0^l (\phi - Fex - \mu + E_x))^{\frac{1}{2}}dx}$$

where $l$ is the width of the effective potential barrier and energies are measured relative to the bottom of the conduction band.

![ET Barrier Effective Potential](image)

**Figure 10. The effective potential for the ET barrier, used in the WKB approximation.**

Though the integral in $D(E_x, V)$ can be solved for a particular $E_x$, a simple, reasonable method is to approximate the integral as the area under a triangular curve. The function is at a maximum at $x = 0$, where the triangle’s "altitude" ($alt.$) is given by

$$alt. = (\phi + \mu - E_x)^{\frac{1}{2}}$$

(17)

The "base" is the distance from $x = 0$ to the location of $(\phi + \mu - Fex - E_x) = 0$:

$$base = x = (\phi + \mu - E_x)/Fe$$

(18)

Therefore the area of the triangular function with dimensions given is found by the elementary formula $A = \frac{1}{2} \times base \times alt.$ and replaces the integral in the exponent of $D(E_x, V)$:

$$D(E_x, V) = f(E_x, V)e^{-2(\frac{2m}{\hbar^2})^{\frac{1}{2}} (\phi + \mu - E_x)^{\frac{1}{2}}/Fe}$$

(19)
Under the assumption that the tunnelling electrons are at the Fermi energy, $D(E_x, V)$ simplifies to

$$D(E_x, V) = f(E_x, V)e^{-\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}(\phi)^{\frac{3}{2}}/F_e}$$

For the Tungsten emitters in this experiment, $\phi = 4.5\, \text{eV}$. This result is useful for understanding transmission and provides a reasonable estimate for the exponential.

2.1.2. Fowler-Nordheim Bessel Function Approach.

The transmission coefficient for the ET barrier can be found directly from Schroedinger’s equation. If the triangle potential barrier begins abruptly at $x = 0$ then there are two regions in which to solve the Schroedinger equation [3]. Inside the potential well, for $x < 0$, the barrier will be zero, and for $x > 0$ it is given by the potential function $V(x) = \phi - Fex$.

Figure 11. The ET barrier with work function $\phi$ as shown above and an electron incident from the left. Here, notice that the potential is modelled as a linearly decreasing function extending to $+\infty$. [3].
\[ \frac{d^2 \psi}{dx^2} - \kappa^2 (\phi - Fex - W) \psi = 0, \quad x > 0 \]  
\[ \frac{d^2 \psi}{dx^2} + \kappa^2 W \psi = 0, \quad x < 0 \]

where \( \kappa \) is defined as
\[ \kappa^2 \equiv \frac{2m}{h^2} \]

where \( h = \frac{\hbar}{2\pi} \), \( \hbar \) is Planck’s constant and \( W \) is the electron energy.

The solution in the region \( x < 0 \) is a combination of left- and right- travelling waves, \( \psi = Ae^{\pm i\kappa x} \). For the region \( x > 0 \), Fowler and Nordheim used the substitution
\[ (-\frac{\phi - W}{Fe} + x)(\kappa^2 Fe)\frac{1}{3} = y \]

By making this substitution using the chain rule, an alternative differential equation with the variable \( y \) is found.
\[ \frac{dy}{dx} = (\kappa^2 Fe)^{\frac{1}{3}} \]

\[ \frac{d^2 \psi}{dy^2} (\kappa^2 Fe)^{\frac{1}{3}} - \kappa^2 (\phi - Fex - W) \psi = 0 \]
\[ \frac{d^2 \psi}{dy^2} - (\kappa^2 Fe)^{\frac{1}{3}} (\frac{\phi - W}{Fe} - x) = 0 \]
\[ \frac{d^2 \psi}{dy^2} + y \psi = 0 \]

This is a differential equation with known solutions, which are related to Bessel and Hankel functions of the second kind with order \( \frac{1}{3} \), \( J_{\pm\frac{1}{3}}^{(2)} \) and \( H_{\frac{1}{3}}^{(2)} \)
\[ \psi = \sqrt{y} J_{\pm\frac{1}{3}}^{(2)} \left( \frac{2}{3} y^2 \right) \]

and
\[ \psi = \sqrt{y} H_{\frac{1}{3}}^{(2)} \left( \frac{2}{3} y^2 \right) \]

The Bessel solutions are real and the Hankel functions are complex. We are interested in a travelling wave solution that oscillates at large \( x \), such as the Hankel function:
\[ \psi = \sqrt{y} H_{\frac{1}{3}}^{(2)} \left( \frac{2}{3} y^2 \right) \approx \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{y^\frac{1}{2}} \frac{1}{(2/3)^{\frac{1}{2}}} e^{-i\left(\frac{3}{2} y^\frac{3}{2} - \frac{2\pi}{12}\right)} \]
The solution for \( x > 0 \) will therefore be in terms of the Hankel function, and the normalized solution for \( x < 0 \) is:

\[
\psi = \frac{1}{W^\frac{1}{2}} \left[ ae^{i\kappa x\sqrt{W}} + a'e^{-i\kappa x\sqrt{W}} \right]
\]

where \( a \) and \( a' \) are the constants to be found, the boundary conditions are satisfied. In particular, \( \psi_+(0) = \psi_-(0) \) and \( \frac{d\psi_+(0)}{dx} = \frac{d\psi_-(0)}{dx} \). The details shown by Fowler and Nordheim are omitted here. Solving for the coefficients \( a \) and \( a' \), the transmission coefficient \( D(W) \) can be found:

\[
D(W) = \frac{4(W(\mu + \phi - W))^{\frac{1}{2}}}{\mu + \phi} e^{-4\kappa(\mu + \phi - W)^{\frac{3}{2}}/3F}
\]

With this more complete equation, the Fowler-Nordheim equation for finding current density as a function of applied field can be found. To do this, recall the electron supply function:

\[
N(v_x, v_y, v_z) dv_x, dv_y, dv_z = 2\left(\frac{m}{h}\right)^3 (1 + e^{(\frac{1}{2}m(v_x^2+v_y^2+v_z^2)-\mu)/kT})^{-1} dv_x dv_y dv_z
\]

Since, as stated earlier, the relevant energy is that associated with the surface normal motion, \( E_x \), integration over \( dv_y \) and \( dv_z \) after changing to polar coordinates in the \( y-z \) plane leads to the desired function:

\[
N(v_x) dv_x = \frac{4\pi m^2}{h^3} dv_x \int_0^\infty (1 + e^{(W-\mu+y)/kT})^{-1} dy
\]

Note that the integration is over an infinite \( y-z \) plane, and the \( \theta \) integration has been evaluated prior to equation 27. The current per unit area will be the product of the supply function, which describes the rate of electron arrival, and the transmission coefficient, which describes the rate of electron tunnelling.

\[
J = \frac{4\pi mekT}{h^3} \int_0^\infty \frac{4(W(\mu + \phi - W)^{\frac{1}{2}} e^{-4\kappa(\mu + \phi - W)^{\frac{3}{2}}/3F}(\int_0^\infty (1 + e^{(W-\mu+y)/kT})^{-1} dy) dW}{\mu + \phi}
\]

\[
= \frac{16\pi me}{(\mu + \phi)h^3} \int_0^\mu W^{\frac{1}{2}}(\mu + \phi - W)^{\frac{1}{2}}(\mu - W)e^{-4\kappa(\mu + \phi - W)^{\frac{3}{2}}/3F} dW
\]

\[
J \simeq \frac{e^{\mu^2}}{2\pi h (\phi + \mu)^{\phi^2}} F^2 e^{-4\kappa\phi^2/3F}
\]

\[
\simeq 6.2 \times 10^6 \frac{\mu^2}{(\phi + \mu)^{\phi^2}} F^2 e^{-2.1 \times 10^8 \phi^2/F}
\]

This is the original Fowler-Nordheim equation. Again assuming a constant emission area and \( F \sim V^2/r \), the linear relationship is:

\[
\ln(I/V^2) \propto 1/V
\]

This relation was discovered experimentally before this result, but Fowler and Nordheim provided the theoretical base for further study of field emission [3].
2.2. A Modified Triangular Barrier with Image Correction.

2.2.1. Gomer Triangular WKB Approximation.

As with the ET barrier, Gomer relies on the WKB approximation and triangular areal approximations to derive the modified FN equation using the SN potential energy barrier [4]. The SN potential energy function accounts for an image effect and, when measured from the ground energy, is given by

\begin{equation}
V(x) = \phi - Fex - \frac{e^2}{4x} \tag{39}
\end{equation}

where \( F \) is in units of V/cm. As can be expected, this is the potential function for the ET barrier with the classical image correction term \(-\frac{e^2}{4x}\). The effective potential in this case is given by

\begin{equation}
p(x) = (\mu - E_x + \phi - Fex - \frac{e^2}{4x})^{1/2} \tag{40}
\end{equation}

![Effective Potential for SN Barrier](image)

**Figure 12.** A plot of the effective potential function for the SN barrier. \( \phi = 4.5 eV \) and \( F = 5 \times 10^{18} V/m \).

The triangle altitude is defined as the function’s maximum, at which its derivative is zero:

\begin{equation}
p_{max} = alt. = (\mu - E_x + \phi - e^{3/2}F^{1/2})^{1/2} \tag{41}
\end{equation}

Assuming that the energy of tunnelling electrons is \( \mu \),

\begin{equation}
alt. = (\phi - e^{3/2}F^{1/2})^{1/2} \tag{42}
\end{equation}

Gomer [4] has given the approximate value for the constants when \( F \) is measured in V/cm as the following:

\begin{equation}
alt. = (\phi - 3.8 \times 10^{-4}F^{1/2})^{1/2} \tag{43}
\end{equation}
Figure 13. The triangular approximation for the effective potential, to be used with the WKB approximation.

Maintain the assumption that $E_x = \mu$. The base is found by the distance from $x = 0$ to x-intercept of $p(x)$:

\begin{align*}
(\phi - Fex - \frac{e^2}{4x})^{\frac{1}{2}} &= 0 \\
(44) &
\end{align*}

\begin{align*}
x &= \frac{\phi + \sqrt{-e^3F + \phi^2}}{2eF} \\
(45) &
\end{align*}

Gomer simplifies this value for $x$

\begin{align*}
\text{base} &= \frac{\phi}{Fe} \\
\end{align*}

Thus the area of the modified barrier is approximately

\begin{align*}
A_{SN} &= \frac{1}{2}(\phi/Fe)(\phi - 3.8 \times 10^{-4}F^{\frac{1}{2}})^{\frac{1}{2}} \\
(46) &
\end{align*}

\begin{align*}
&= \frac{1}{2}(\phi^{\frac{3}{2}}/Fe)(1 - 3.8 \times 10^{-4}F^{\frac{1}{2}}/\phi)^{\frac{1}{2}} \\
(47) &
\end{align*}

\begin{align*}
D(E_x, V) &= f(E_x, V)e^{-\left(\frac{2\mu}{\hbar^2}\right)^{\frac{1}{2}}(\phi^{\frac{3}{2}}/F)(1 - 3.8 \times 10^{-4}F^{\frac{1}{2}}/\phi)^{\frac{1}{2}}} \\
(48) &
\end{align*}

Recall the ET barrier approximate solution:

\begin{align*}
D(E_x, V) &= f(E_x, V)e^{-\left(\frac{2\mu}{\hbar^2}\right)^{\frac{1}{2}}(\phi^{\frac{3}{2}}/F)/Fe} \\
\end{align*}

The transmission coefficient for the SN barrier, and as a result the FN equation, varies by a factor, call it $\alpha$, where

\begin{align*}
\alpha &= \sqrt{1 - y} \\
(49) &
\end{align*}
\[ y = 3.8 \times 10^{-4} F \frac{1}{2} / \phi \]

Typically \( \alpha \) has values between 1 and 0 and \( y \) between 0 and 1 respectively. The resulting Fowler-Nordheim equation can be written in terms of \( r \), using \( F = \frac{V}{r} \), and allowing \( k \) to incorporate the constants and any field modifications for a non-ideal sphere. Note that the leading constant differs slightly from our result because it arises from more rigorous calculations, but the effect is the same. This relation is used below to find \( r \).

\[ \ln(I/V^2) = \ln(a) - 6.8 \times 10^7 \alpha kr \phi^{\frac{3}{2}} / V \]

where \( a \) is a known constant.

### 2.3. The Murphy-Good Modified FN Equation.

The modified FN equation may be less popular in field emission experiments because of the complicated nature of the final result \[7\]. However, it remains one of the most accurate derivations in FN theory. Included below is a sketch of the derivation as given in the original publication \[19\].

If the potential energy barrier with image correction, or SN barrier, begins at the surface of a metal at \( x = 0 \) and the lowest energy in a metal is set at \(-W_a\), then \( V(x) \) is given by

\[ V(x) = -\frac{e^2}{4x} - eFx, \quad x > 0 \]

\[ V(x) = -W_a, \quad x < 0 \]

Likewise, the total energy of an electron in the tunnelling direction is given by

\[ E_x = \frac{p^2(x)}{2m} + V(x) \]

where \( E_x \) is the part of the energy for motion normal to the surface. It is assumed that near \( x = 0 \) the function \( V(x) \) is smooth. The transmission coefficient can be given by a slightly different form of the WKB approximation used previously:

\[ D(F,E_x) = \frac{1}{1 + e^{-\frac{2i}{\hbar} \int_{x_1}^{x_2} p(x) dx}} \]

From the energy equation for an electron, \( p(x) \) can be found:

\[ p(x) = \sqrt{2m(W + \frac{e^2}{4x} + eFx)} \]

In the bound region, \( x_1 \) and \( x_2 \) are the classical turning points and are chosen so that \( x_1 < x_2 \) and as a result the exponent in the equation for \( p(x) \) is real. Using the above expression for \( p(x) \) in the WKB approximation,

\[ -\frac{3i}{\hbar} \int_{x_1}^{x_2} p(x) dx \]

\[ D(F,W) = 1 + e^{-\frac{2i}{\hbar} \int_{x_1}^{x_2} \sqrt{2m[E_x + \frac{e^2}{4x} + eFx]} dx} \]
A substitution can be made so that $D(F, E_x)$ is rewritten

\begin{equation}
D(F, E_x) = 1 + e^{\frac{4}{3} \sqrt{2} \left( \frac{F \hbar^4}{m^2 e^5} \right)^{-\frac{1}{2}} y^{-\frac{3}{2}} v(y)}
\end{equation}

\begin{equation}
y = \frac{\sqrt{e^3 F}}{|E_x|},
\end{equation}

\begin{equation}
v(y) = -\frac{3i}{4\sqrt{2}} \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \left[ \rho - 2 + y^2 \rho^{-1} \right]^\frac{1}{2} d\rho
\end{equation}

Here $y$ is a variable specific to Murphy-Good [19] and unrelated to the previous result. In practice, $y$ is between 0 and 1 [19]. $v(y)$ is given by

\begin{equation}
v(y) = 2^{-\frac{1}{2}} (1 + a)^{\frac{1}{2}} (E[(2a)^{\frac{1}{2}}/(1 + a)^{\frac{1}{2}}] - (1 - a) K[(2a)^{\frac{1}{2}}/(1 + a)^{\frac{1}{2}}]
\end{equation}

where

\begin{equation}
K[k] = \int_0^{\pi/2} (1 - k^2 \sin^2(\theta))^{-\frac{1}{2}} d\theta
\end{equation}

\begin{equation}
E[k] = \int_0^{\pi/2} (1 - k^2 \sin^2(\theta))^{\frac{1}{2}} d\theta
\end{equation}

are the common elliptic integrals and $a$ is given by

\begin{equation}
a = (1 - y^2)^{\frac{1}{2}}
\end{equation}

Note that $y$ is field-dependent. $v(y)$ is then determined by a known value of $F$ or through experiment. The numerical solutions for $v(y)$ for realistic values of $y$ published by Burgess, Kroemer and Houston [6] compared to Gomer’s values for $\alpha$ and are shown below.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$v(y)$</th>
<th>$y$</th>
<th>$v(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>0.55</td>
<td>0.6351</td>
</tr>
<tr>
<td>0.05</td>
<td>0.9948</td>
<td>0.6</td>
<td>0.5768</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9817</td>
<td>0.65</td>
<td>0.5152</td>
</tr>
<tr>
<td>0.15</td>
<td>0.9622</td>
<td>0.7</td>
<td>0.4505</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9370</td>
<td>0.75</td>
<td>0.3825</td>
</tr>
<tr>
<td>0.25</td>
<td>0.9068</td>
<td>0.8</td>
<td>0.3117</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8718</td>
<td>0.85</td>
<td>0.2379</td>
</tr>
<tr>
<td>0.35</td>
<td>0.8323</td>
<td>0.9</td>
<td>0.1613</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7888</td>
<td>0.95</td>
<td>0.0820</td>
</tr>
<tr>
<td>0.45</td>
<td>0.7413</td>
<td>1.0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6900</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Using this transmission coefficient, Murphy and Good obtained a complete modified FN equation. Note that \( v(y) \) acts as a correction factor similar to Gomer’s \( \alpha \).

\[
J = \frac{F^2}{16\pi^2 \phi [t(y)]^2} e^{-\frac{4\sqrt{2} e^2}{3} \frac{v(y)}{\phi}}
\]

(66)

\[
t(y) = v(y) - \frac{2}{3} y \frac{dv(y)}{dy}
\]

(67)

The modified FN equation is able to more accurately predict observed emission currents for a known emission area. It can be linearized to obtain the familiar form:

\[
\ln \left( \frac{I}{V^2} \right) \propto \frac{1}{V}
\]

(68)

Experimental data is fit to this linear relationship. The relationship does not require the knowledge of the emission area and therefore allows the correction factor \( v(y) \equiv \alpha \) to be determined empirically.

2.4. Hyperboloidal Models for Nanotips.

In the past two decades field emission research has taken a new direction, focusing on fabricating nanoscale tips and emitting from alternative materials like carbon nanotubes. The creation of nanoscale emitters prompted a new look at the FN equation. For example, in He et al. 1991, the transmission coefficient is derived for potential barriers of conical, parabolic, hyperboloidal and sphere on cone tip geometry [8] in an attempt to better describe the effect of the electric field on electron tunnelling for sharp Spindt cathodes [8].

Such emitter arrays originated in the 1970s and have been used in technologies such as electronic displays and ionization sources. The tip radii are typically on the order of 1-100 nanometers. The Spindt array is defined by its ability to contain a high density per area of emitters and the closeness of the counterelectrode to the emitter. Due to the counterelectrode closeness, the threshold voltage for field emission is lower and higher emission currents can be achieved.

Recently, research such as He et al. [20] has assumed that sharp, Spindt cathode-type emitters are hyperboloidal rather than spherical. The tip potential energy is assumed to be proportional to the geometry and also hyperboloidal. This hyperboloidal potential energy curve is plotted in 3D the cartesian coordinate system and then mapped onto an ellipsoidal coordinate system. Once the image term is accounted for and the resulting potential discovered, the Schroedinger equation is solved using elliptical coordinates to obtain a FN-type equation. The potential barrier used is shown in Figure 15, with radius 10nm and bias voltage 20V.
Figure 14. A silicon Spindt array of nano-emitters, scale 1µm. The sharp emitters are located close to the counterelectrode [12].

Figure 15. Potential energy barrier as a function of distance z from apex. $E_F$, the Fermi energy, has a value of $-\phi = -4.5eV$ for Tungsten. From top, dotted lines show the bias field and the potential barrier for a cone emitter model. Solid lines show the bias field and potential barrier for the hyperboloidal model [8].

The resulting nonlinear empirical relationship for a hyperboloidal is given, but was not used in our data analysis:

$$\ln\left(\frac{I}{V^2}\right) \propto \frac{1}{V} + \frac{1}{V^2}$$
Yuasa et. al. 2003 [13] produced a more simple hyperboloidal calculation, neglecting the image charge altogether. The potential energy barrier used is shown in figure 16.

\[
U(z, \rho = 0) \quad L \sim 5 \text{cm} \\
\text{eV} \sim 3 \text{keV}
\]

\[
U(z, \rho = 0) \quad \sim 1 \text{nm}
\]

\[
\phi \approx 4.5 \text{eV}, \text{ for Tungsten [13]}
\]

From the FN equation given by this potential, Yuasa has derived the linear model

\[
\ln \frac{I}{V^3} \propto \frac{1}{V}
\]

The Yuasa et. al. fit has been included in analysis.

3. Experimental Setup

3.1. Chemical Etching.

Tungsten emission tips were etched from 0.5mm- to 2mm- diameter wire in a 1.5M Sodium Hydroxide solution (6) using a procedure for Scanning Tunnelling Microscope (STM) tips. Etched tips were then rinsed with clean Methyl Alcohol, Acetone, and deionized water.

The Tektronix CPS250 power supply (2) and the carbon cathode (4) were connected (3,5) to form the etching circuit. Around the meniscus formed at the wire-solution interface, the wire material was preferentially etched as shown in Figure 18, causing a curved narrow region. When the wire was etched through at the narrowest region, the trailing end fell and the etching current dropped to almost 0mA, as observed on a Tektronix CDM250 digital multimeter (1). This time was recorded and the applied voltage was terminated, leaving a sharp tip. Typical etching times were between one and five minutes.
Figure 17. *A diagram of the tip etching setup.*

Figure 18. *Preferential etching of the tip caused by the meniscus at the wire-solution interface. When the wire is completely etched, a sharp tip remains* [17].
Etching variations were tested to obtain a range in emitter radii. The etching circuit variables are shown in the table below for each tip. There was not a strong correlation between these variables and the tip radius.

**Table 2.** Etching variables. Note: The radius is not listed due to a scarcity of SEM-imaged successful emitters.

<table>
<thead>
<tr>
<th>Tip</th>
<th>Wire diameter (mm)</th>
<th>Molarity (M)</th>
<th>Etching Voltage (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>0.05</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>1.10</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>2.10</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>1.20</td>
<td>0.2</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>2.20</td>
<td>0.2</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>1.111613</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>2.111613</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>3.111613</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>4.111613</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>1.012114</td>
<td>0.1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>2.012114</td>
<td>0.1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3.012114</td>
<td>0.1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>1.012214</td>
<td>0.1</td>
<td>2.5</td>
<td>10</td>
</tr>
<tr>
<td>2.012214</td>
<td>0.1</td>
<td>2.5</td>
<td>10</td>
</tr>
<tr>
<td>1.012914</td>
<td>0.1</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>2.012914</td>
<td>0.1</td>
<td>1.5</td>
<td>11</td>
</tr>
<tr>
<td>3.012914</td>
<td>0.1</td>
<td>1.5</td>
<td>12</td>
</tr>
<tr>
<td>4.012914</td>
<td>0.1</td>
<td>1.5</td>
<td>13</td>
</tr>
<tr>
<td>5.012914</td>
<td>0.1</td>
<td>1.5</td>
<td>14</td>
</tr>
</tbody>
</table>
After etching, tips were examined under an optical microscope. Even tips that appear sharp at this magnitude may not succeed in emitting if there are microscopic surface features or contamination on the tip. In addition, gas molecules in the vacuum chamber will become adsorbed on the tip. See 3.2 for methods of adsorbed gas molecule removal.

**Figure 19.** An etched Tungsten emitter shown under an optical microscope. The tip’s apparent sharpness is not a singular indicator of emission success.

3.2. **Field Emission and Data Collection.** Field emission was conducted in a stainless steel vacuum chamber at $10^{-7}$ Torr. An Edwards backing pump (4) and Pfeiffer-Balzers TPH 240 turbo pump (2) were used to obtain the vacuum pressure. The emission tip was secured in a copper holder by stainless steel washers (8, inset) and attached to a high voltage feedthrough inside a 5-way cross attachment (1). We arranged a phosphor screen (7), attached via a feedthrough to an electrometer, perpendicular to the direction of electron emission in order to detect the emission current.
With the tip secured, the vacuum pump-down time was typically 5-24 hours. This period enabled the removal of unwanted gases in the chamber. To remove additional adsorbed
gases from the tip, the Bertran Associates Inc. Model 205B-10R high voltage power supply (5) was turned to a low voltage (<1kV) and run for 15-30 minutes. After this period, the voltage was varied by 50-500V from about 1kV to 3kV. A Granville-Phillips 330 Ionization Gauge (3) and Controller (6) measured the vacuum pressure during experiment.

We ran Vernier hardware through the electrometer to obtain 30 seconds of current data, where it could be stored and analyzed with LoggerPro software. The image shows a sample data collection in LoggerPro and the software’s statistical analysis. The mean emission current in each collection was recorded as one sample point.

![Sample current vs. applied voltage (I vs. V) data set from field emission trials.](image)

**Figure 22.** Sample current vs. applied voltage (I vs. V) data set from field emission trials.

### 3.3. Scanning Electron Microscopy.

The University of Minnesota Characterization Facility staff imaged field emission tips using a Hitachi S-4700 Scanning Electron Microscope (SEM) at an accelerating voltage of 10kV. From May 2013 to January 2014, six SEM images of successful emitting tips were obtained. These images allow us to directly observe possible emitting surfaces on a 50 nanometer scale. Additional images of unsuccessful, melted or damaged tips were obtained but were not used in this study. (Appendix A [7]).
4. Data Analysis

4.1. Radius Determination.

The linear modified FN equation provides a relationship between the emission current, the applied voltage and the radius of an approximately spherical emitter. From Gomer [4], this equation is written:

\[
\ln\left(\frac{I}{V^2}\right) = \ln(a) - b' \frac{\phi^2}{V}
\]

(71)

\[
b' = 6.8 \times 10^7 \alpha kr
\]

(72)

where \(a\) is a known constant, \(I\) is the electron emission current, and \(V\) is the applied voltage.

A correction factor \(k\), has been added to account for non spherical emitters such that \(F_{max} = \frac{V}{kr}\):

\[
F = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r^2} = \frac{V}{r}
\]

\(F\)

(73)

\[
F_{max} = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{kr^2} = \frac{V}{kr}
\]

Experimental data can be plotted as \(\ln(I/V^2)\) vs. \(1/V\) and fit to the linear equation above.
The slope of such a plot is equivalent to $-b'\phi^2$, where

$$(74) \quad b' = 6.8 \times 10^7 \alpha kr$$

In practice the $y$ value ranges between 0 and 1, and so $\alpha$ ranges between 1 and 0. To find a lower bound on $kr$, let $\alpha = 1$ and solve for $kr$ from the slope of the plot.

$$(75) \quad kr_{min} = \frac{\text{Slope}}{6.8 \times 10^7 \alpha \phi^2}$$

A more accurate result for $kr$ can be obtained by using $kr_{min}$ to find $F_{max,avg} = \frac{V_{avg}}{kr_{min}}$.

New values for $y$ and $\alpha$ are found:

$$(76) \quad y = 3.8 \times 10^{-4} (F_{max,avg})^{\frac{1}{2}} / \phi$$

$$(77) \quad \alpha = \sqrt{1 - y}$$

which in turn changes the values of $y$ and $\alpha$. Iteration of the procedure causes $kr$ to converge on a single value [4]. The iteration process has been coded in Mathematica and is shown below.
Figure 23. The Mathematica code used for iteration to find the most accurate $kr$ value.

It is common practice to let $k = 5$ [4] so that $r$ can be discovered from an appropriate $kr$ value.

Suggested emission tips were modelled directly using a circle model fit over SEM images. The chosen emission surfaces are likely FN emitters due to characteristic extrusion, curvature and size, but emission from these surfaces has not been confirmed. The radius can be extracted from these model fits.
The two radius determination methods were compared directly. For radius determination from minimum and iterative methods, \( k = 5 \) was used. We then recalculated \( k \) from the measured SEM radius values, finding \( k \) values typically 2 or 3 times larger than the standard \( k = 5 \) [4]. This new \( k \) value may indicate that standard analysis of Scanning Tunnelling Microscopy (STM) radii significantly underestimate tip sharpness.

<table>
<thead>
<tr>
<th>Tip</th>
<th>SEM radius (nm)</th>
<th>Iterative radius (nm)</th>
<th>Minimum radius (nm)</th>
<th>( k ) value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>68.8</td>
<td>113.2</td>
<td>77.91</td>
<td>8.227</td>
</tr>
<tr>
<td>2.1</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>11.32</td>
</tr>
<tr>
<td>1.1</td>
<td>19.33</td>
<td>71.15</td>
<td>44.49</td>
<td>18.40</td>
</tr>
<tr>
<td>1.1</td>
<td>-</td>
<td>57.7</td>
<td>43</td>
<td>14.92</td>
</tr>
<tr>
<td>1.1</td>
<td>-</td>
<td>50.22</td>
<td>34.08</td>
<td>12.99</td>
</tr>
<tr>
<td>1.2</td>
<td>35.25</td>
<td>81.5</td>
<td>55.92</td>
<td>11.56</td>
</tr>
<tr>
<td>2.012214</td>
<td>20.25</td>
<td>imaginary</td>
<td>16.11</td>
<td>1.488*</td>
</tr>
<tr>
<td>4.012914</td>
<td>67</td>
<td>imaginary</td>
<td>19.94</td>
<td>3.977*</td>
</tr>
</tbody>
</table>

### 4.2. Tip Field Modelling.

The Mathematica hyperbola tip model fits for three successful tips from summer 2013 are shown in Table ???. Each tip is labelled by \( N.LL \), where \( N \) is the tip identity number and \( 0.LL \) is the diameter in millimeters of the Tungsten wire before etching. For example, 2.10 is the second etched tip of initial wire diameter 0.10mm. Not shown, this modelling process was applied to all suggested emission surfaces near a tip apex.
Table 5. Hyperbolic fits. Top: Tip 1.10, Tip 2.10. Bottom: Tip 1.20

Two additional images of successful tips were obtained in February 2014 and the model fits are shown in Table 6. Each tip is labelled by N.MMDDYY, where N is the tip identity number, M is the month, D the date and Y the year the tip was etched.
From these hyperbolic models, a spherical fit with identical curvature at the tip apex can be recovered. Below is a sample Mathematica code showing this procedure. The equations for hyperbola and sphere are given in cartesian coordinates and $a$ and $b$ are hyperbolic parameters that were determined from the hyperbolic model fit. This process can be applied to every tip by assigning the appropriate values to $a$ and $b$.

In future research, the electric field for each geometry could be determined using a finite element analysis program like FlexPDE. A sample code for FlexPDE Student Version 6 is shown in figures 25 and 26. This version has inadequate computational power and our results are preliminary. For future research, we suggest analysis in a program or version with sufficient computational capability.
Figure 24. A Mathematica program matched a circle to the curvature at the tip of a hyperbola. This method could be used in future research to compare the theoretical electric field for complex geometries to the traditional spherical assumption.
Figure 25. Sample code for calculating the electric field from a hyperboloidal tip in FlexPDE Student6.

Figure 26. A vector plot of the electric field created by FlexPDE. The field values are not necessarily accurate and axes scale is arbitrary.
4.3. **Linear Models for FN Equation.** Thirteen field emission data sets were fit to linear models derived from three forms of the FN equation. Each model is shown below.

4.3.1. **FN Linear Plot.**

The most familiar linear model for CFE research is derived from the FN equation:

\[ \ln(I/V^2) \propto 1/V \]  

(78)

This linear relationship was first observed by experimentalists like Schottky [2] before Fowler and Nordheim published a more thorough theoretical result [3]. We have fit all obtained data sets to this linear model.

![FN Empirical Linear Model, Gomer](image)

**Figure 27. FN CFE fit to the original FN linear model [3]**

4.3.2. **Yuasa et. al. Hyperboloidal Model.**

Yuasa et. al. 2003 [13] derived the potential barrier for a hyperboloid using an ellipsoidal coordinate system. Unlike early hyperboloidal tip theory by He et al [20], Yuasa et. al. did not include an image charge in the potential barrier. The Yuasa hyperboloidal linear model is similar to the FN linear model.

\[ \ln(I/V^3) \propto 1/V \]  

(79)

Our data was plotted according to Yuasa et. al. and the empirical linear relationship obtained. For nearly all data fit to this model, the $R^2$ measure of deviation from fit is lower for the model of Yuasa et. al. than for the standard FN linear model, indicating that it does not fit our data as accurately.
4.3.3. Forbes $\kappa$ Fit.

Forbes 2008 suggested a linear model of the form [16]:

$$I = CV^\kappa e^{-B/V}$$

where $B$ and $C$ are constants. This equation should be questioned because it assumes that $V$ scales according to a power law. For example, the Forbes derivation for $\kappa$ is given and
fit to our experiment CFE data:

$$ln(I) = ln(C/(\frac{1}{V})^{\kappa}) - B/V$$  \hspace{1cm} (81)

$$ln(I) = ln(C) - \kappa ln(\frac{1}{V}) - B/V$$  \hspace{1cm} (82)

Taking the negative of the derivative with respect to $1/V$

$$-\frac{d(ln(I))}{d(1/V)} = \kappa V + B$$  \hspace{1cm} (83)

We have found an average rate of change by fitting a linear regression to 3-5 data points and recording each slope as a single value of $\frac{d(ln(I))}{d(1/V)}$. Plotted against $V$, these values should provide a linear data set whose slope is $\kappa$.

The data below indicates that this method does not provide a reasonable value of $\kappa$. Forbes states the expected value of $\kappa = 1.2$, whereas we see $\kappa \sim 10^{-2}$.

**Figure 30.** The slope of a $ln(I)$ vs. $1/V$ plot was used to represent $\frac{d(ln(I))}{d(1/V)}$, Tip 1.05.

**Figure 31.** A plot of $\frac{d(ln(I))}{d(1/V)}$ vs $V$, Tip 1.05.
Figure 32. The slope of a $\ln(I)$ vs. $1/V$ plot was used to represent $\frac{d\ln(I)}{d(1/V)}$, Tip 1.1.

Forbes [16] assumes that the voltage $V$ scales simply according to a power law. However if, as we suspect, the applied field and resulting voltage act as a more complicated function $f(V)$, then it is easy to show that Forbes’ method will not produce a reliable $\kappa$ value.

\[(84)\]
\[I = Cf(V)^\kappa e^{-B/V}\]

where $f$ is an arbitrary function of $V$. Continue by deriving Forbes relation for $\kappa$ from this equation.

\[(85)\]
\[\ln(I) = \ln(C) + \kappa \ln(f(V)) - B/V\]

\[(86)\]
\[\frac{d\ln(I)}{d(1/V)} = -\kappa \frac{d(f(V))}{d(1/V)} \frac{d(\ln(f(V)))}{d(1/V)} + B\]

Clearly, the slope of such a plot is not strictly given by a constant $\kappa$. Further evidence for the inadequacy of Forbes’ assumption for experimental data is shown in figures 31 and 33.
Traditional field emission models refer to large, sphere-on-a-cone emitter geometries. In modern field emission it is common to have sharp tips with radii on the nanometer scale, causing a strong field effect on tunnelling electrons. The aim of this study was to compare field emission data analyzed by traditional methods to SEM images of emission tips and more recent methods of FE analysis. This comparison illuminated the shortcomings of traditional radius determination methods and confirmed the traditional linear plot model over newer approaches.

We have examined the iterative method \[4\] for finding the emitter radius. Using SEM radius measurements, we have shown that the standard value \(k = 5\) for the iterative method significantly overpredicts radius values for emitters. This overprediction is a concern because it provides inaccurate indications of STM and other FE tips’ performance in both experiment and FE technology.

While recent publications have taken a theoretical approach to improving FN FE analysis for sharp tips, we suggest discovering the electric field directly using images of successful field emission tips. As shown in this paper, the tip parameters can be extracted through high resolution imaging such as by Scanning Electron Microscope. A finite element analysis program then allows calculation of the electric field from these real, often complicated tip geometries. The electric field can be correlated to theoretical empirical models relying on field emission data in order to infer which models are most physically accurate.

6. Summary

In this experiment, 11 Tungsten field emission tips of radius \(\sim 20 - 50\)nm were electrochemically etched and successfully tested for FN field emission in a vacuum chamber. Of the successful emitters, clean SEM images of 5 tips were obtained for modelling purposes.

The radius of tips was determined by an iterative method based on an approximation of the modified FN equation. This iterative radius was compared to the radius measured by circular fit modelling of SEM images, allowing the discovery of a larger \(k\) correction value.

Hyperbolic models were also fit to SEM images of emitters. From these models, circles with identical curvature at the apex can be discovered. A sample FlexPDE code is provided for reference.

Our experimental field emission current and applied voltage data was plotted according to three linear models derived from FN-type equations. The first plot supported the empirical linear relation \(\ln(I/V^2) \propto 1/V\) which can be found from both the original and modified FN equation \([3, 19, 4]\). Second, the data was plotted according to \(\ln(I/V^3) \propto 1/V\), from the Yuasa et. al. \([13]\) FN-type equation for a hyperboloidal emitter barrier in elliptical coordinates without image term. Last, the Forbes relation \(\ln(I/V^\kappa) \propto 1/V\) was used to plot our experimental data. The \(\kappa\) values were of the order of \(10^{-2}\), much smaller than 1.2 predicted by Forbes. An argument is made against scaling \(V\) in the FN equation according to a power law. \([16]\)
In the age of sharper and increasingly unconventional emitters, standard FN emitters provide a base for comparing new and old theoretical and experimental models. Future research could include improved electric field computations for experimentally-based tip geometries, better tip etching procedures and variation in the range of radii with an emphasis on testing new hyperboloidal models, and a deeper analysis and comparison of empirical radius extraction methods.

References

7. Appendix A.

7.1. FN Linear Plot Models and Emission Tip Images.

Table 7. Tip 1.20, Summer 2013.

Table 8. Tip 1.10, Summer 2013.
Table 9. Tip 2.10, Summer 2013.

Table 10. Tip 3.20, Summer 2013.
No emission.
Table 11. Tip 2.012214.

Table 12. Tip 4.012914.
Table 13. Tip 5.012914.
No emission.

Table 14. Tip 2.111613.
No SEM image.
Table 15. Tip 3.111613.
No SEM image.

Table 16. Tip 4.111613.
No SEM image.
Table 17. Tip 1.012114.
No SEM image.

Table 18. Tip 3.012114.
No SEM image.